

Notes 24 largely plagiarized by %khc

This material is optional. It will be a preview for those of you who wish to take ee123.

1 Filter Design

There are a number of ad-hoc ways to convert a given CT filter specification into a DT filter specification:

- impulse invariance
- step invariance
- backward difference
- rectangular integration
- trapezoidal integration/bilinear transform

In the impulse invariance technique, you take the CT impulse response $h(t)$ and sample it to produce $h[n]$:

$$h[n] = Th(t)|_{t=nT}$$

Then you transform $h[n]$ to get $H(z)$. Note that in impulse invariance, you need to normalize by T to ensure that $H(e^{j\omega T})$ is approximately the same as $H(\omega)$.

What in the frequency domain are you doing when you sample in the time domain? Recall that sampling a CT signal $x(t)$ produces a DT signal $x[n]$; the DTFT of $x[n]$, denoted by $X(e^{j\omega T})$, is a function of the Fourier transform of $x(t)$, denoted $X(\omega)$. In particular:

$$X(e^{j\omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\frac{2\pi}{T})$$

Because $h[n] = Th(t)|_{t=nT}$,

$$\begin{aligned} H(e^{j\omega T}) &= T \frac{1}{T} \sum_{k=-\infty}^{\infty} H(\omega - k\frac{2\pi}{T}) \\ &= \sum_{k=-\infty}^{\infty} H(\omega - k\frac{2\pi}{T}) \end{aligned}$$

In other words, if we graph $H(e^{j\omega T})$ against ω , we end up with copies of the original CT spectrum, copied at multiples of $\frac{2\pi}{T}$. Most of the time, $H(\omega)$ will have really long tails as ω goes to infinity, and the copies of $H(\omega)$ will alias, overlapping each other and distorting the $H(e^{j\omega T})$. Increasing the sample rate by lowering the sampling period T will act to remove some of this distortion [cf sampling/aliasing in notes23].

In the step invariance technique, instead of sampling the impulse response, you sample the step response:

$$y_{step}[n] = y_{step}(t)|_{t=nT}$$

Z transforming $y_{step}[n]$ gives $Y_{step}(z)$. Now $y_{step}[n] = h[n] * u[n]$, so $Y_{step}(z) = H(z)\mathcal{Z}[u[n]]$; with $\mathcal{Z}[u[n]] = \frac{1}{1-z^{-1}}$, we can solve for $H(z)$:

$$H(z) = (1 - z^{-1})Y_{step}(z)$$

It is not obvious what normalization should be performed, if any at all. For now, i will say none is necessary, although if you are given a particular relationship that is required to hold [eg $H(\omega)|_{\omega=0} = H(e^{j\omega T})|_{\omega=0}$], then you should make sure that it does.

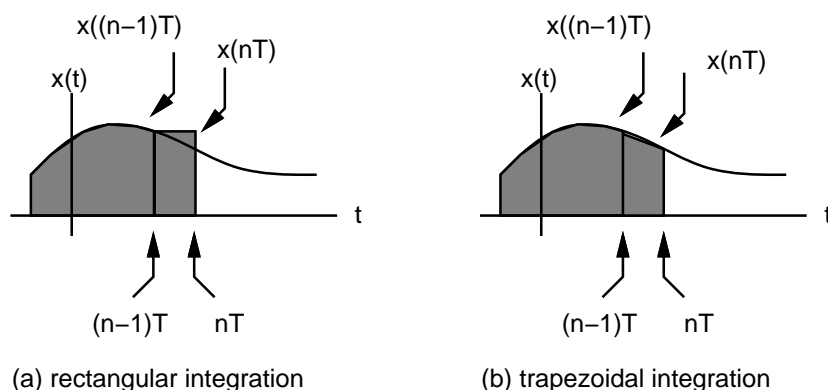


Figure 1: Two integration schemes.

One possible approximation for the first derivative of a CT signal $x(t)$ is to take the difference of two successive samples of $x(t)$, and divide by the time interval between those samples:

$$\dot{x}(t) \approx \frac{1}{T}(x[n] - x[n-1])$$

This approximation is called a backward difference. Since the equivalent of taking a derivative in CT is multiplying by s in the s domain, we can relate s and z by

$$s = \frac{1}{T}(1 - z^{-1})$$

since the Z transform of $\frac{1}{T}(x[n] - x[n-1])$ is $\frac{1}{T}(1 - z^{-1})$. So if you are given $H(s)$, you can construct $H(z)$ by making the substitution $s = \frac{1}{T}(1 - z^{-1})$.

Consider Figure 1(a). In the rectangular integration technique, if $y((n-1)T)$ is the area underneath the curve at time $t = (n-1)T$, the area underneath the curve at time $t = nT$ can be approximated by a box with height $x((n-1)T)$ and width T :

$$y(nT) \approx y((n-1)T) + Tx((n-1)T)$$

So

$$\frac{Y(z)}{X(z)} = \frac{Tz^{-1}}{1 - z^{-1}}$$

. Integration corresponds to $\frac{1}{s}$ in the s domain; in the z domain, we have $\frac{Tz^{-1}}{1 - z^{-1}}$. So if you are given $H(s)$, you can construct $H(z)$ by making the substitution $s = \frac{1 - z^{-1}}{T}$.

Trapezoidal integration is another integration scheme, but instead of approximating the area underneath the curve by a box, a trapezoid is used instead, as in Figure 1(b). If $y((n-1)T)$ is the area underneath the curve at time $t = (n-1)T$, the area underneath the curve at time $t = nT$ is then:

$$y(nT) \approx y((n-1)T) + \frac{T}{2}[x((n-1)T) + x(nT)]$$

So

$$\frac{Y(z)}{X(z)} = \frac{T}{2} \frac{1 + z^{-1}}{1 - z^{-1}}$$

Integration corresponds to $\frac{1}{s}$ in the s domain; in the z domain, we have $\frac{T}{2} \frac{1 + z^{-1}}{1 - z^{-1}}$. So if you are given $H(s)$, you can construct $H(z)$ by making the substitution $s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$.

The reason why we care about the trapezoidal integration is that it can be interpreted as a DT filter design technique in which no aliasing occurs. To see this, consider the mapping $s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$, evaluated in terms of frequency. With

$s = j\omega_c$ and $z = e^{j\omega_d T}$, where ω_c denotes “continuous” ω and ω_d denotes “discrete” ω :

$$\begin{aligned} j\omega_c &= \frac{2}{T} \frac{1 - e^{-j\omega_d T}}{1 + e^{-j\omega_d T}} \\ &= \frac{2}{T} \frac{e^{j\omega_d T/2} - e^{-j\omega_d T/2}}{e^{j\omega_d T/2} + e^{-j\omega_d T/2}} \\ &= \frac{2}{T} \frac{2j \sin \omega_d \frac{T}{2}}{2 \cos \omega_d \frac{T}{2}} \\ &= \frac{2}{T} j \tan \omega_d \frac{T}{2} \\ \omega_c &= \frac{2}{T} \tan \omega_d \frac{T}{2} \end{aligned}$$

What does this do? Note that as ω_d ranges from $-\frac{\pi}{T}$ to $\frac{\pi}{T}$, ω_c sweeps from $-\infty$ to ∞ . So if we replace the ω in $H(\omega)$ by $\frac{2}{T} \tan \omega_d \frac{T}{2}$ to get $H(e^{j\omega_d T})$, we end up warping $H(\omega)$ so that it fits in the range from $-\frac{\pi}{T}$ to $\frac{\pi}{T}$.

Note also that $\frac{2}{T} \tan \omega_d \frac{T}{2}$ is periodic with period $\frac{2\pi}{T}$, so that each time you go through one period of $H(e^{j\omega_d T})$, you end up traversing $H(\omega)$ from $-\infty$ to ∞ . The bottom line is that there is no aliasing of the $H(\omega)$, but that the $H(\omega)$ gets warped to fit in the interval from $-\frac{\pi}{T}$ to $\frac{\pi}{T}$.

2 A Note on Aliasing

A number of people may still be confused by the term “aliasing”. In the context of sampling a signal $x(t)$ to produce $X(e^{j\omega T})$, aliasing refers to sampling below the Nyquist rate [twice the maximum frequency in $x(t)$], causing the copies of the spectrum of $x(t)$ to overlap and get messed up. In the context of constructing a DT filter from a CT filter specification, aliasing refers to having copies of $H(\omega)$ overlap and perturbing your resulting $H(e^{j\omega T})$.

For a DT system, if $x[n]$ is the input, $h[n]$ is the impulse response, and $y[n]$ is the output, we already know that:

$$y[n] = h[n] * x[n]$$

In terms of the Z transform:

$$Y(z) = H(z)X(z)$$

In terms of the DTFT [obtained by replacing z by $e^{j\omega T}$]:

$$Y(e^{j\omega T}) = H(e^{j\omega T})X(e^{j\omega T})$$

So, when we say that there is no aliasing when we use the bilinear transform, we *don't* mean that the input doesn't get aliased. If $x[n]$ came from sampling some $x(t)$, and you didn't sample it at above the Nyquist rate, your $X(e^{j\omega T})$ has been messed up already, and unless you have some additional information you're not telling me, there is no hope. Run down the hall, screaming your head off and tearing the rest of your hair out.

What we actually mean when there is no aliasing with the bilinear transform is that the entire $H(\omega)$ gets warped and squished into an interval from $-\frac{\pi}{T}$ to $\frac{\pi}{T}$.

3 Looking Around

One more topic: the discrete Fourier transform (DFT).¹ Make sure that you know what the differences are among:

- the Fourier series,
- the Fourier transform,
- the Laplace transform,

¹i didn't make up these names, so stop flaming me if you are having difficulty telling the DTFT and the DFT apart.

- the Z transform,
- and the discrete-time Fourier transform (DTFT).

Now would be a good time to start reviewing for the final.