Notes 07 largely plagiarized by %khc

1 Warning

This set of notes covers the Fourier transform. However, i probably won't talk about everything here in section; instead i will highlight important properties or give random examples.[1] You are advised to consult your lecture notes, this set of notes, and your textbook, since you are responsible for everything.

2 Fourier Transform

We have already seen the Fourier transform; an input $x(t) = e^{j\omega t}$ into an LTI system with impulse response $h(t)$ gives an output $y(t)$:

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\ &= \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}h(\tau)d\tau \\ &= e^{j\omega t}\int_{-\infty}^{\infty} e^{-j\omega\tau}h(\tau)d\tau \\ &= e^{j\omega t}H(\omega) \end{aligned}$$

where $H(\omega)$ is defined as $\int_{-\infty}^{\infty} e^{-j\omega\tau}h(\tau)d\tau$. $H(\omega)$ is the Fourier transform of $h(t)$.

So what? How can we interpret the Fourier transform, other than as more annoying math that we have to remember?

Before, we noted that the Fourier series decomposes a *periodic* signal into *discrete* frequencies, each a multiple of some basic frequency. Now, we have the Fourier transform, which can decompose an *aperiodic* signal also into its component frequencies; however, these frequencies are not multiples of some frequency, but range from $\omega = -\infty$ to $\omega = \infty$.

To see this, we can start from the formula for FS coefficients, and then take the limit as $T \to \infty$ (thereby increasing the period until we effectively have only one period to consider):

$$\begin{aligned} X(\omega) &\stackrel{\Delta}{=} \lim_{T\to\infty} Ta_k \\ &= \lim_{T\to\infty} \int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \end{aligned}$$

Note that we have $\omega \stackrel{\Delta}{=} k\omega_0$. As $T \to \infty$, $\omega_0 = \frac{2\pi}{T}$ becomes infinitesimally small, but we can always find some k such that $\omega = k\omega_0$ for the ω in which we are interested.

3 FT Properties

Linearity The Fourier transform is a linear operation. This means that if $x(t) \leftrightarrow X(\omega)$ and $y(t) \leftrightarrow Y(\omega)$ then

- $ax(t) \leftrightarrow aX(\omega)$.

[1] Randomness is good.

- $x(t) + y(t) \leftrightarrow X(\omega) + Y(\omega)$.

Utility value: very high.

Conjugate symmetry If $x(t)$ is real, $X(w) = X^*(-w)$.
As with FS:

$$\begin{aligned} X^*(-\omega) &= \int_{-\infty}^{\infty} x^*(t)e^{+j(-\omega)t}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= X(\omega) \end{aligned}$$

This also implies that $\mathcal{R}e\ (X(\omega))$ and $|X(\omega)|$ are even functions, and $\mathcal{I}m\ (X(\omega))$ and $\angle(X(\omega))$ are odd functions.
Utility value: checking work.

Even and odd symmetry If $x(t)$ is real and even, $X(\omega)$ is real and even.
If $x(t)$ is real and odd, $X(\omega)$ is imaginary and odd.
Utility value: checking work.

Scale change $x(at) \leftrightarrow \frac{1}{|a|}X(\frac{\omega}{a})$
Contraction in time ($a < 0$) causes expansion in frequency. Expansion in time ($a > 0$) causes contraction in frequency. Think Heisenberg.
Utility value: testing situation.

Time shift $x(t - T) \leftrightarrow X(\omega)e^{-j\omega T}$
A delay or advance does not change the magnitude of the Fourier transform, but the phase changes by $-\omega T$. If $\angle(X(\omega))$ is linear with ω, this is termed "linear phase", a phrase is used when referring to filters. If you are interesting in passing a certain band of frequencies without modifying them too much, you would like to have linear phase in that band, because linear phase corresponds to an advance or delay in time. Nonlinear phase would tend to phase shift some frequencies more than others, causing very interesting output.
Utility value: very high.

Modulation $e^{j\Omega t}x(t) \leftrightarrow X(\omega - \Omega)$
Shifting in frequency is modulation by a complex exponential in time. Why would we want to shift in frequency? Well, if everybody used -20 kHz to 20 kHz for communication, nothing much would get done. So if we assign bands of frequencies to everybody who wants to communicate, and modulated our signals with complex exponentials at frequency Ω, we would end up centering our original -20 kHz to 20 kHz around Ω. If we allocating frequencies correctly so that there were sufficient guard bands between consecutive blocks of frequency, we could all just get along.

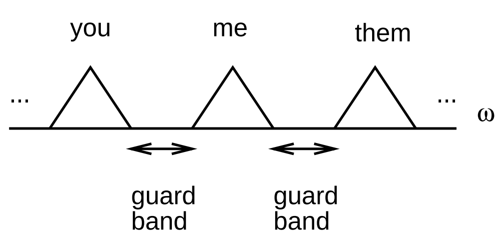

Figure 1: Guard bands.

In this country, frequency allocation is done by the Federal Communications Commission (FCC)[2].
Utility value: very high.

[2] i had a brief run-in with them a couple of years ago, but that's another story [available during office hours, if desired].

Convolution in time is multiplication in frequency. $x(t) * y(t) \leftrightarrow X(\omega)Y(\omega)$
This gives us another way to do those annoying convolution integrals. Of course, don't get carried away and transform everything to avoid convolution— it may actually be a lot easier at some points in time to stay in the time domain (eg convolution in time with a series of impulses).
Utility value: very high.

Multiplication in time is convolution in frequency except for a factor of $\frac{1}{2\pi}$. $x(t)y(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * Y(\omega)$
Try not to forget the $\frac{1}{2\pi}$, the source of many lost points.[3] Convince yourself that modulation is a special case of this.
Utility value: very high.

Duality If $g(t) \leftrightarrow f(\omega)$ then $f(t) \leftrightarrow 2\pi g(-\omega)$.
Note the negative sign and the factor of 2π, another source of lost points.
Utility value: very high, especially for deriving transforms not in the table.

Integration $\int_{\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
The delta function accounts for the DC in x(t).
Utility value: testing situation.

Differentiation $\frac{d}{dt}x \leftrightarrow j\omega X(\omega)$
This gives us another way to solve LDEs. see below.
Utility value: very high.
Exercise Prove these properties.
Exercise Compare these properties to those of the Fourier series. Hmmm, some of them look pretty much the same, except that $k\omega_0$ gets replaced by ω...

4 Fourier Transform Symmetry Properties

If $x(t) \leftrightarrow X(\omega)$, what happens to $x(-t)$, $x^*(t)$, and $x^*(-t)$?

$$
\begin{aligned}
\mathcal{F}[x(-t)] &= \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt \\
&= -\int_{\infty}^{-\infty} x(\tau)e^{j\omega t}d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)e^{-j(-\omega)t}d\tau \\
&= X(-\omega) \\
\mathcal{F}[x^*(t)] &= \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt \\
&= [\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt]^* \\
&= [\int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt]^* \\
&= X^*(-\omega) \\
\mathcal{F}[x^*(-t)] &= \int_{-\infty}^{\infty} x^*(-t)e^{-j\omega t}dt \\
&= -\int_{\infty}^{-\infty} x^*(\tau)e^{j\omega t}d\tau
\end{aligned}
$$

[3] If you do leave it out, you've just created free energy [see Parseval's identity for energy signals].

$$
\begin{aligned}
&= [\int_{-\infty}^{\infty} x(\tau)e^{-j\omega t}d\tau]^* \\
&= X^*(\omega)
\end{aligned}
$$

We also need to extend the notion of evenness and oddness to complex-valued functions:

$$
\begin{aligned}
\mathcal{CS}\ [x(t)] &= \frac{1}{2}[x(t) + x^*(-t)] \\
\mathcal{CAS}\ [x(t)] &= \frac{1}{2}[x(t) - x^*(-t)]
\end{aligned}
$$

Given the above and some knowledge of $x(t)$, we can actually say some useful things about $X(\omega)$.

$$
\begin{aligned}
\mathcal{R}e\ [x(t)] &= \frac{1}{2}[x(t) + x^*(t)] \\
&\leftrightarrow \frac{1}{2}[X(\omega) + X^*(-\omega)] \\
&\leftrightarrow \mathcal{CS}\ [X(\omega)] \\
\mathcal{CS}\ [x(t)] &= \frac{1}{2}[x(t) + x^*(-t)] \\
&\leftrightarrow \frac{1}{2}[X(\omega) + X^*(\omega)] \\
&\leftrightarrow \mathcal{R}e\ [X(\omega)]
\end{aligned}
$$

Exercise Prove $\mathcal{I}m\ [x(t)] = \frac{1}{2}[x(t) - x^*(t)] \leftrightarrow \mathcal{CAS}\ [X(\omega)] = \frac{1}{2}[x(t) - x^*(-t)]$ and $\mathcal{CAS}\ [x(t)] \leftrightarrow \mathcal{I}m\ [X(\omega)]$.
Exercise Compare to the Fourier series symmetry properties. Notice something.

5 Fun FT facts

The DC level of a signal in the frequency domain is $X(0)$.

$$X(0) = \int_{-\infty}^{\infty} x(t)dt$$

The value $x(0)$ can be recovered by integrating over the corresponding Fourier transform.

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega$$

The area under $\frac{\sin ax}{bx}$ is the area of the inscribed triangle under the main lobe of the function. This area is $\frac{\pi}{b}$.
The Fourier transform of a gaussian, a function of the form $e^{-\frac{(x-a)^2}{b}}$, is a gaussian (of the same form).
The Fourier transform of an impulse train is an impulse train.
The Fourier transform of 0 is 0.

6 Useful FTs

Delta function $\delta(t)$:

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} \delta(t)dt \\
&= 1
\end{aligned}
$$

Constant 1**:** $\delta(t) \leftrightarrow 1$, so $1 \leftrightarrow 2\pi\delta(\omega)$ by duality.

Signum function sgn (t)**:** The signum function is defined as:

$$\text{sgn}\,(t) = \begin{cases} +1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases}$$

Note the behavior at $t = 0$.

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} \text{sgn}\,(t) e^{-j\omega t} dt \\ &= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} e^{-\epsilon|t|} \text{sgn}\,(t) e^{-j\omega t} dt \\ &= \lim_{\epsilon \to 0} [-\int_{-\infty}^{0} e^{\epsilon t - j\omega t} dt + \int_{0}^{\infty} e^{-\epsilon t - j\omega t} dt] \\ &= \lim_{\epsilon \to 0} [-\frac{1}{\epsilon - j\omega} e^{(\epsilon - j\omega)t}|_{-\infty}^{0} - \frac{1}{\epsilon + j\omega} e^{-(\epsilon + j\omega)t}|_{0}^{\infty}] \\ &= \lim_{\epsilon \to 0} [-\frac{1}{\epsilon - j\omega} + \frac{1}{\epsilon + j\omega}] \\ &= \lim_{\epsilon \to 0} -\frac{2j\omega}{\epsilon^2 + \omega^2} \\ &= -\frac{2j\omega}{\omega^2} \\ &= \frac{2}{j\omega} \end{aligned}$$

Unit step $u(t)$**:**

$$\begin{aligned} X(\omega) &= \mathcal{F}(u(t)) \\ &= \mathcal{F}(\frac{1}{2} + \frac{1}{2}\text{sgn}\,(t)) \\ &= \pi\delta(\omega) + \frac{1}{j\omega} \end{aligned}$$

Doublet $\dot{\delta}(t)$**:** $\delta(t) \leftrightarrow 1$, so $\dot{\delta}(t) \leftrightarrow j\omega$ by differentiation property.

Complex exponential $e^{j\Omega t}$**:** $1 \leftrightarrow 2\pi\delta(\omega)$ so $e^{j\Omega t} \leftrightarrow 2\pi\delta(\omega - \Omega)$ by modulation. This can also be done using duality on the time shift property.

Sine $\sin \Omega t$**:**

$$\begin{aligned} X(\omega) &= \mathcal{F}(\sin \Omega t) \\ &= \mathcal{F}(\frac{1}{2j}[e^{j\Omega t} - e^{-j\Omega t}]) \\ &= -j\pi[\delta(\omega - \Omega) - \delta(\omega + \Omega)] \end{aligned}$$

Since sine is real and odd, its Fourier transform is purely imaginary.

Cosine $\cos \Omega t$**:**

$$\begin{aligned} X(\omega) &= \mathcal{F}(\cos \Omega t) \\ &= \mathcal{F}(\frac{1}{2}[e^{j\Omega t} + e^{-j\Omega t}]) \\ &= \pi[\delta(\omega - \Omega) + \delta(\omega + \Omega)] \end{aligned}$$

Since cosine is real and even, its Fourier transform is purely real.

Pulse $\Pi(t)$**:** If a pulse is defined as:

$$x(t) = \begin{cases} 1 & \text{if } -T_1 < t < T_1 \\ 0 & \text{elsewhere} \end{cases}$$

then its transform is:

$$\begin{aligned} X(\omega) &= \int_{-T_1}^{T_1} e^{-j\omega t} dt \\ &= -\frac{1}{j\omega} e^{-j\omega t}|_{-T_1}^{T_1} \\ &= -\frac{1}{j\omega}[e^{-j\omega T_1} - e^{+j\omega T_1}] \\ &= \frac{2 \sin \omega T_1}{\omega} \end{aligned}$$

Exponential $e^{-at}u(t)$**:**

$$\begin{aligned} X(\omega) &= \int_0^{\infty} e^{-at-j\omega t} dt \\ &= -\frac{1}{j\omega + a} e^{-(j\omega+a)t}|_0^{\infty} \\ &= \frac{1}{j\omega + a} \end{aligned}$$

assuming $\mathcal{R}e\ a > 0$.

Any Fourier series

$$\begin{aligned} X(\omega) &= \mathcal{F}(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}) \\ &= \sum_{k=-\infty}^{\infty} a_k \mathcal{F}(e^{jk\omega_0 t}) \\ &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) \end{aligned}$$

7 Fourier Transform Interpretation

Now that we have seen the Fourier transform, how exactly can we interpret what it's doing? Well, one way is to say that the Fourier transform of $x(t)$ gives the "frequency content" of $x(t)$.

Consider the complex exponential $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$. $\mathcal{F}[e^{j\omega_0 t}]$ contains only one frequency at $\omega = \omega_0$.

Consider the constant $1 \leftrightarrow 2\pi\delta(\omega)$. $\mathcal{F}[1]$ contains only one frequency at $\omega = 0$, corresponding to DC.

Consider the cosine $cos\omega_0 t \leftrightarrow \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$. $\mathcal{F}[cos\omega_0 t]$ contains two frequencies, one at $-\omega_0$ and the other at ω_0. i will now blithely dodge the concept of explaining negative frequency, except to say that negative frequencies can be thought of as a mathematical construct in order to understand sines and cosines.

8 Time Limited and Band Limited Signals

If $x(t)$ is time limited, then $X(\omega)$ has infinite bandwidth. To see this, consider a signal $x_0(t)\Pi(\frac{t}{a})$, limited in time to duration a. $x_0(t)$ is multiplied by a pulse in time, so in frequency $X_0(\omega)$ is convolved with a sinc. Generally, convolution of "well-behaved" signals (eg nothing with doublets or other derivatives of δ) with a signal of infinite extent is going to give you something of infinite extent. So we in the frequency domain, $X(\omega)$ will not go to zero for all ω greater than some ω_0. Thus, our signal is not bandlimited.
Exercise Show that if $X(\omega)$ is bandlimited, then $x(t)$ is not of finite duration.

9 A Consistency Check

We should be able to use the integration and differentiation properties of the Fourier transform to prove that:

- the derivative of $\delta(t)$ is the doublet.
- the integral of the doublet is $\delta(t)$.
- the integral of $\delta(t)$ is $u(t)$.
- the derivative of $u(t)$ is $\delta(t)$.

To prove these statements, first tranform the function into the frequency domain, apply the appropriate FT property, and tranform back into the time domain.
Exercises Prove these.

10 Solving LDEs

The general form of a linear differential equation is:

$$\sum_{n=0}^{N} a_n \frac{d^n}{dt^n} y = \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} x$$

This would be a flaming pain to solve for large M and N. Fortunately, we have the Fourier transform differentiation property. Through repeated application of this property, we obtain $\frac{d^n}{dt^n} y \leftrightarrow (j\omega)^n Y(\omega)$. So, assuming that the system is at rest (zero state, or zero initial conditions), we then have:

$$\sum_{n=0}^{N} a_n (j\omega)^n Y(\omega) = \sum_{m=0}^{M} b_m (j\omega)^m X(\omega)$$

We have reduced a differential equation to an algebraic one. This is a feature.

In fact, we can solve this equation for $\frac{Y(\omega)}{X(\omega)}$:

$$\frac{Y(\omega)}{X(\omega)} = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n}$$

If $x(t)$ is the input to an LTI system and $h(t)$ is its impulse response, then its output is $y(t) = x(t) * h(t)$. If $X(\omega) = \mathcal{F}(x(t))$, $H(\omega) = \mathcal{F}(h(t))$, and $Y(\omega) = \mathcal{F}(y(t))$, then using the fact that convolution in the time domain is multiplication in the frequency domain $Y(\omega) = X(\omega)H(\omega)$.

What does this have to do with the math at the beginning of the section? We know $Y(\omega) = X(\omega)H(\omega)$, or $H(\omega) = \frac{Y(\omega)}{X(\omega)}$. This then gives us:

$$H(\omega) = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n}$$

Compare this to your results from problem set 4, problem 9, and realize that you found another way to say the same thing.

So, if you give me an LDE describing the operation of some LTI system, i can give you its $H(\omega)$ and corresponding impulse response $h(t)$. In fact, for $x(t) = \delta(t)$, $X(\omega) = 1$, so $Y(\omega) = H(\omega)$.

Once we have $H(\omega)$, we can then determine $Y(\omega)$ for any $X(\omega)$. So if you give me any input $x(t)$, i can determine its Fourier transform $X(\omega)$, find $Y(\omega)$, and then inverse Fourier transform to get $y(t)$. No convolution! What a feature!

11 Inverse Fourier Transform

Inverse FTs are extremely painful to do, since they require some familiarity with complex analysis. Instead, we will use the time-honored method of table lookup. This is why familiarity with the transforms listed above in the "Useful FTs" section is useful.

Of course, it would be too much to expect every function we deal with to be easily found in a table. If certain conditions are met, we can use partial fraction expansion to reduce the function into simpler ones which we hopefully can find in our tables for inverse transforming.

12 Partial Fraction Expansion

The Heaviside method is a short-cut for determining the partial fraction expansion of a rational function of the form $\frac{f(x)}{g(x)}$, where the degree of $f(x)$ is less than the degree of $g(x)$ and $g(x)$ can be factored into linear terms (no powers of x greater than 1). Assume that $g(x)$ can be written as $(x - r_1)(x - r_2)\cdots(x - r_N)$ where the roots of $g(x)$ are distinct. Then:

$$\begin{aligned}
\frac{f(x)}{g(x)} &= \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_N)} \\
&= \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \cdots + \frac{A_N}{x - r_N}
\end{aligned}$$

where we have blithely assumed that we could do the partial fraction expansion (not exactly mathematically robust, but so is life).

To find a formula for A_1:

$$\begin{aligned}
\frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_N)} &= \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \cdots + \frac{A_N}{x - r_N} \\
\frac{f(x)}{(x - r_2)\cdots(x - r_N)} &= A_1 + \frac{A_2(x - r_1)}{x - r_2} + \cdots + \frac{A_N(x - r_1)}{x - r_N} \\
\frac{f(x)}{(x - r_2)\cdots(x - r_N)}|_{x=r_1} &= [A_1 + \frac{A_2(x - r_1)}{x - r_2} + \cdots + \frac{A_N(x - r_1)}{x - r_N}]|_{x=r_1} \\
\frac{f(x)}{(x - r_2)\cdots(x - r_N)}|_{x=r_1} &= A_1
\end{aligned}$$

Formulas for the other A are found similarly.

An example:

$$\begin{aligned}
\frac{1}{(x+5)(x-6)} &= \frac{A_1}{x+5} + \frac{A_2}{x-6} \\
A_1 &= (x+5)\frac{1}{(x+5)(x-6)}|_{x=-5} = -\frac{1}{11} \\
A_2 &= (x-6)\frac{1}{(x+5)(x-6)}|_{x=6} = \frac{1}{11} \\
\frac{1}{(x+5)(x-6)} &= \frac{-1/11}{x+5} + \frac{1/11}{x-6}
\end{aligned}$$

What if the roots of $g(x)$ are not distinct? We can then use a differentiation trick. Example:

$$
\begin{aligned}
\frac{5x+2}{(x-1)^2} &= \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} \\
5x+2 &= A_1(x-1) + A_2 \\
5x &= A_1 x \Rightarrow A_1 = 5 \ \text{ differentiating once wrt } x \\
(5x+2)|_{x=1} &= [A_1(x-1)+A_2]|_{x=1} \Rightarrow A_2 = 7 \\
\frac{5x+2}{(x-1)^2} &= \frac{5}{x-1} + \frac{7}{(x-1)^2}
\end{aligned}
$$

13 Putting It All Together

Suppose we have a system described by the linear differential equation

$$y'' - y' - 30y = x$$

So by the above reasoning, its impulse response $h(t)$ is:

$$
\begin{aligned}
(j\omega)^2 Y(\omega) - (j\omega)Y(\omega) - 30Y(\omega) &= X(\omega) \\
\frac{Y(\omega)}{X(\omega)} &= \frac{1}{(j\omega+5)(j\omega-6)} \\
H(\omega) &= \frac{1}{(j\omega+5)(j\omega-6)} \\
h(t) &= \mathcal{F}[\frac{-1/11}{j\omega+5} + \frac{1/11}{j\omega-6}] \\
&= -\frac{1}{11}e^{-5t}u(t) + \frac{1}{11}e^{6t}u(t)
\end{aligned}
$$

Note that the system is not BIBO stable.
Exercise Why?

14 A Look Around

We have covered quite a bit of Fourier transform material. You should be familiar with all of it. Since i have finite time in section, i'm going to have to depend on your not-as-finite time in order to review all this. Please do so.