Notes 00 largely plagiarized by %khc

1 Signal Operations

We have seen a number of operations that can be performed on signals:

- scale change
- time reversal
- time shift
- linear combination

For fun, let's try constructing $x(a-t)$.

To check your work, verify a few interesting points in $x(a-t)$. First, let's define $y(t)$ equal to $x(a-t)$. Note that $y(t)$ at $t = a+1$ should be equal to $x(t)$ at $t = -1$; this can be seen by looking at Figure 1. Since $y(t) = x(a-t)$, at $t = a+1$ we have $y(a+1) = x(a-(a+1)) = x(-1)$, as desired.

Why did we bother doing this? Well, we're going to need it for the sifting and convolution integrals.

2 Periodic Signals

A signal is periodic if it satisfies the equation: $x(t+T) = x(t)$. Of the signals we will study in the course, the sinusoids are going to be encountered most frequently.

If we have two sinusoids $\sin ax$ and $\sin bx$, is their sum periodic? What restrictions must we make on a and b?

$$\begin{aligned} f(x) &= \sin ax + \sin bx \\ f(x+T) &= \sin a(x+T) + \sin b(x+T) \\ &= \sin ax \cos aT + \cos ax \sin aT + \sin bx \cos bT + \cos bx \sin bT \end{aligned}$$

If we wish $f(x) = f(x+T)$, then $\cos aT = \cos bT = 1$ and $\sin aT = \sin bT = 0$. So $aT = 2\pi m$ and $bT = 2\pi n$, where m and n are integers. Since the T are the same, we then have $m/n = a/b$. Because m and n are integers, this implies that a/b is rational. Does this make sense? Say we have two sinusoids, one with period 1 and the other with period π. Are the zero crossings ever going to match up? No.

Of course, this works for cosines and combinations of sines and cosines.

Exercise Verify this.

3 Complex Exponentials

We have seen Euler's identity before (although perhaps in a previous life). This comes from considering the Taylor series for $\cos x$, $\sin x$, and e^x.

From calculus, a Taylor series about some point $x = a$ is defined as:

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots \\ &= \sum_{k=0}^{\infty} \frac{f^{(k)}(x-a)^k}{k!} \end{aligned}$$

If we apply the above formula to $\cos x$, $\sin x$, and e^x, all expanded about $x = 0$, we get:

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \ldots \end{aligned}$$

In particular, if we look for e^{jx}, we obtain:

$$\begin{aligned} e^{jx} &= 1 + jx + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \frac{(jx)^4}{4!} + \frac{(jx)^5}{5!} \ldots \\ &= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots) + j(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots) \\ &= \cos x + j \sin x \end{aligned}$$

An exceedingly cool thing is to note that

$$e^{j\pi} = -1$$

This is similar to going on vacation in East Kalamazoo, MI, walking by a cafe, and seeing four friends sitting there.

This allows us to find alternate ways of representing $\cos x$ and $\sin x$. Adding $e^{jx} = \cos x + j \sin x$ and $e^{-jx} = \cos x - j \sin x$ gives:

$$\cos x = \frac{1}{2}[e^{jx} + e^{-jx}]$$

Subtracting the two gives:

$$\sin x = \frac{1}{2j}[e^{jx} - e^{-jx}]$$

The complex exponentials $e^{j\omega t}$ are a subset of the exponentials e^{st} where $s = \sigma + j\omega$. If we examine $\mathcal{R}e\ e^{st}$:

$$\begin{aligned} \mathcal{R}e\ e^{st} &= \mathcal{R}e\ [e^{\sigma t} e^{j\omega t}] \\ &= \mathcal{R}e\ [e^{\sigma t}(\cos \omega t + j \sin \omega t)] \\ &= e^{\sigma t} \cos \omega t \end{aligned}$$

We could have three different types of functions depending on the value of σ:

- a exponentially growing sinusoid if $\sigma > 0$.
- a not-so-interesting sinusoid if $\sigma = 0$.
- an exponentially decaying sinusoid if $\sigma < 0$.

If you've ever used ATT's calling cards, the "bong" that you hear when you are prompted for your calling card number is an exponentially decaying sinusoid.

Exercise Examine $\mathcal{I}m\ e^{st}$. Make some useful statements.

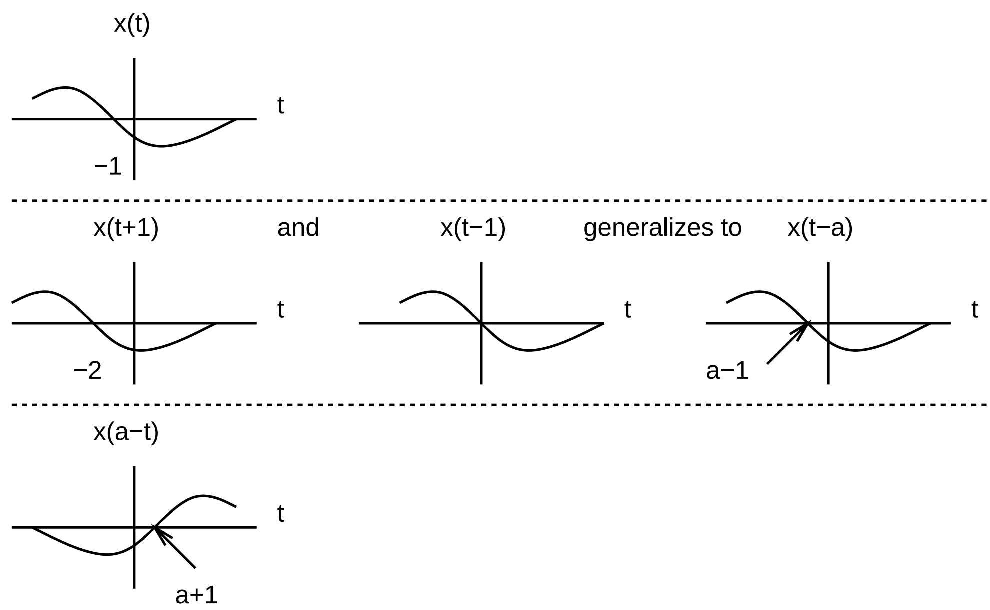
Why bother studying these also? Well, complex exponentials are going to show up quite a lot as:

- eigenfunctions for linear, time invariant systems.
- modulation.
- solutions to second order linear differential equations that describe second order systems.

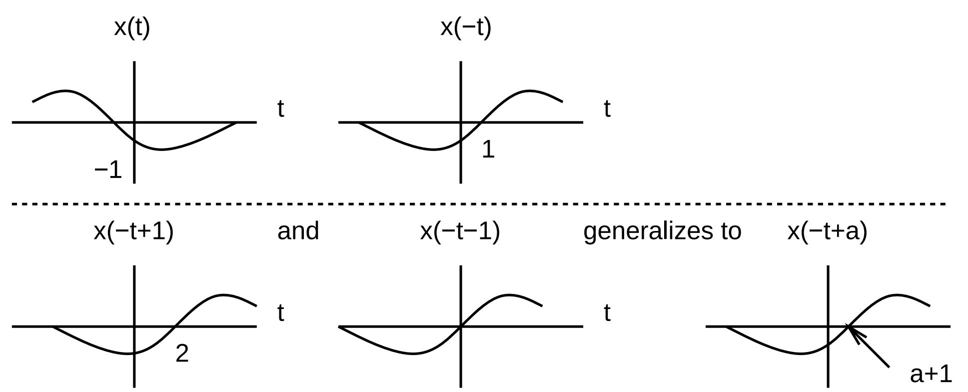
If nothing else, make sure that you are familiar with the alternate definitions for cosine and sine, since they will keep showing up.

4 A Look Ahead

We will be needing signal operations for the convolution integral. Make sure that you know how to generate $f(a - t)$, with a constant, t variable.



(a) one method of constructing x(a−t)



(b) another method of constructing x(a−t)

Figure 1: Constructing $x(a-t)$.