Professor Fearing EECS120/Problem Set 8 v 1.0 Fall 2016

Due at 4 pm, Fri. Oct. 21 in HW box under stairs (1st floor Cory)MT Wed Oct 26 410-6 pm.
Closed book, closed notes. Two sides 8.5x11 inch formula sheet. Coverage PS1-PS8, Lectures 1-14.

Note: $\Pi(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2})$, and $comb(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$.

GSI: Ming Jin.

Problem 1 LTI Properties (21 pts)

[15 pts] Classify the following systems, with input $x(t)$ and output $y(t)$. In each column, write "yes", "no", or "?" if the property is not decidable with the given information. (+1 for correct, 0 for blank, -0.5 for incorrect).

System	Causal	Linear	Time-invariant
a. $y(t) = x(t)\cos(2\pi t)$	yes	yes	no
b. $y(t) = x(t) * u(t-2)$	yes	yes	yes
c. $y(t) = 3x(t+1) + 1$	no	no	yes
d. $y(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)d\tau$	no	no	no
e. $y(t) = x(t) - \frac{1}{2}\frac{dy(t)}{dt}$	yes	NO	NO

1(c) not linear since $x(t)=0$ gives $y(t)=$

[6 pts] Two of the systems above (a,b,c,d,e) are not BIBO stable. Note below which systems are not BIBO stable, and then find a bounded input $x(t)$ which gives rise to an unbounded output $y(t)$ for each of these systems.

System 1: b

Bounded input $x(t) =$ 1

$$y(t) = 1 * u(t-2) = \int_{-\infty}^{\infty} u(\tau-2)d\tau \rightarrow \infty$$

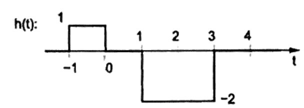
System 2: d

Bounded input $x(t) =$ 1

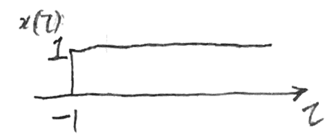
$$y(t) = \int_{-\infty}^{\infty} 1\, d\tau = +\infty$$

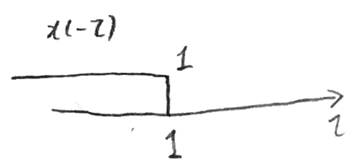
Note: Reasons for 1e being not linear or time-invariant is that we need to consider the initial conditions of the system, which is not captured in the input x(t). For instance, if the initial condition is y(0) = 1, then even though we have x(t)=0, we still have y(t) to be non-zero. and delaying x(t) does not necessarily delay the output y(t).

[6 pts] e. An LTI system has impulse response $h(t)$ as shown below:

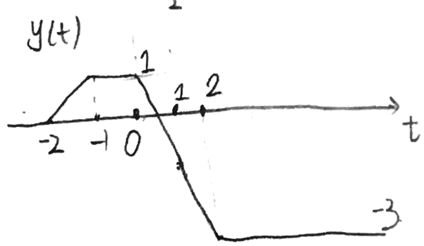


Given input $x(t) = u(t+1)$. Sketch the output $y(t)$ on the grid below, noting key times and amplitudes.





Answer:



Graphical convolution

2. (20 pts) Consider an LTI system (with input $x[n]$ and output $y[n]$defined by the difference equation:

$$y[n] = -0.25x[n] + 0.5x[n-1] - 0.25x[n-2]$$

a. Determine if this system is causal and/or stable.
b. Determine the frequency response $H(e^{j\omega})$ and sketch its magnitude $|H(e^{j\omega})|$ as a function of ω. Determine the type of filter (low pass, highpass, bandpass, or bandstop) realized by this system.
c. Determine whether this system is linear phase.
d. Draw a block diagrma implementing this system with delay, summation, and multiplication blocks.

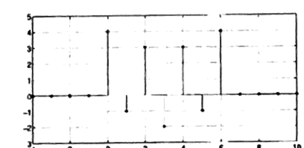
Given the sequence x[n] depicted below, determine the following:

e. $X(e^{j0})$

f. $X(e^{j\pi})$

g. $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$

h. $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$



a) Since $y[n]$ does not depend on future inputs, it is causal.

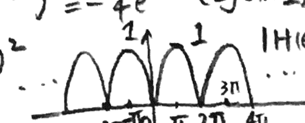
Since $y[n] = x[n] * \underbrace{(-0.25\delta[n] + 0.5\delta[n-1] - 0.25\delta[n-2])}_{h[n]}$

and $\sum_{n=-\infty}^{\infty} |h[n]| = 1$ is abs. summable, the system is (BIBO) stable

(b). $H(e^{jw}) = -0.25 + 0.5e^{-jw} - 0.25e^{-j2w}$

$= -\frac{1}{4}(1-e^{-jw})^2 = -\frac{1}{4}e^{-jw}(2j\sin\frac{w}{2})^2 = e^{-jw}(\sin\frac{w}{2})^2$

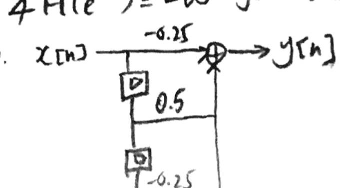
$|H(e^{jw})| = (\sin\frac{w}{2})^2$



This is high-pass filter.

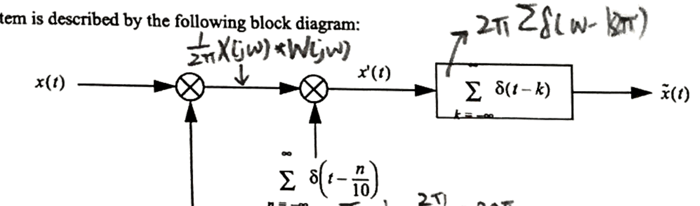
(c). $\measuredangle H(e^{jw}) = -w$ yes it is linear phase

(d).



Problem 7 (35 points)

A system is described by the following block diagram:

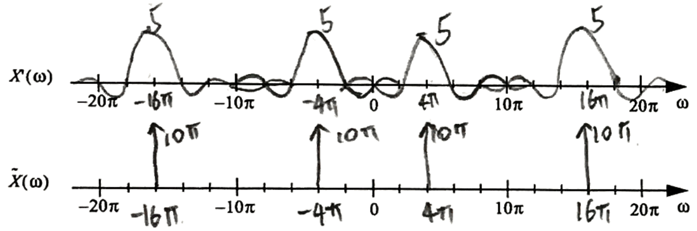


$T_s = \frac{1}{10}$, $\frac{2\pi}{T_s} = 20\pi$

$$w(t) = \begin{cases} 1 & \text{if } -.5 \le t < .5 \\ 0 & \text{otherwise} \end{cases}$$

$W(jw) = \frac{2\sin\frac{w}{2}}{w}$

[30 pts.] a) Let $x(t) = \cos 4\pi t$. Sketch $X'(\omega)$ and $\tilde{X}(\omega)$, labelling peak magnitude, zero crossing(s), and spacing. (Hint: $X'(\omega)$ and $\tilde{X}(\omega)$ should be real.)



$$\mathcal{F}\{x(t)\cdot w(t)\} = \frac{1}{2\pi}X(jw) * W(jw) = \frac{1}{2\pi}\left[\pi(\delta(w-4\pi)+\delta(w+4\pi))\right] * W(jw)$$

$$= \frac{1}{2}W(j(w-4\pi)) + \frac{1}{2}W(j(w+4\pi))$$

where $W(jw)$ is sinc w/ zero crossing at 2π multiples, except for $w=0$ which has value of 1.

$$\mathcal{F}\{x'(t)\} = \mathcal{F}\{(x(t)\cdot w(t))\cdot \sum\delta(t-\tfrac{n}{10})\} = \frac{1}{2\pi}\mathcal{F}\{x(t)w(t)\} * \frac{2\pi}{\frac{1}{10}}\sum\delta(w-k20\pi)$$

$$= 5\sum_{k=-\infty}^{\infty}\left[W(j(w-4\pi-k20\pi)) + W(j(w+4\pi-k20\pi))\right]$$

$\tilde{X}(jw)$ are simply samples of $X'(jw)$ at 2π multiples.

[5 pts.] b) What is the relationship between $\tilde{X}(\omega)$ and the 10 point DFT of $x[n] = X[k]$ (where $x[n] = x(0)\ldots x(.9)$)? Explain why. (What is the effect of not shifting the window $w(t)$ by $T/2$?)

The new window $w'(t) = w(t-\frac{1}{2})$, so $\mathcal{F}\{w'(t)\} = W(jw)e^{-jw\frac{1}{2}}$

Using the procedure as above, this window corresponds to FT:

$$\bar{X}'(jw) = 5\sum_{k=-\infty}^{\infty}\left[W'(j(w-4\pi-k20\pi)) + W'(j(w+4\pi-k20\pi))\right]$$

$$= 5\sum_{k=-\infty}^{\infty}\left[W(j(w-4\pi-k20\pi))e^{-j\frac{1}{2}w} + W(j(w+4\pi-k20\pi))e^{-j\frac{1}{2}w}\right]$$

$$\left(\text{Since } e^{-j\frac{1}{2}(w-4\pi-k20\pi)} = e^{-j\frac{1}{2}w}\underbrace{e^{j2\pi}e^{jk10\pi}}_{1} = e^{-j\frac{1}{2}w}\right)$$

$$= 5e^{-j\frac{1}{2}w}X'(jw)$$

From the DFT H.O., we know that

$$X[k] = \frac{T_0}{2\pi}\,\text{area}\left[\bar{X}'(jk\tfrac{2\pi}{T_0})\right], \text{ w/ } T_0 = 1,$$

and $\bar{X}'(jk\frac{2\pi}{T_0}) = 5e^{-j\frac{1}{2}k\frac{2\pi}{T_0}}X'(jk\frac{2\pi}{T_0})$

$$= 5e^{-jk\pi}X'(jk2\pi)$$

where $\bar{X}'(jw)$ is found above

Therefore $X[k] = \frac{5e^{-jk\pi}}{2\pi}\,\text{area}[X'(jk2\pi)]$

However, since $X'(jk2\pi)$ is non-zero only when $k = \pm 2, \pm 8$, at which point $e^{-jk\pi} = 1$, therefore, in this particular case $X[k] = \frac{5}{2\pi}\,\text{area}[X'(jk2\pi)]$, the shifting window has no effect on DFT.

prob 2 continue.

e. $X(e^{j0}) = \sum x[n] = 4-1+3-2+3-1+4 = 10$

f. $X(e^{j\pi}) = \sum x[n]\, e^{-j\pi n} = \sum x[n]\, (-1)^n$

$= 4+1+3+2+3+1+4 = 18$

g $\int_{-\pi}^{\pi} X(e^{jw})\, dw = \int_{-\pi}^{\pi} X(e^{jw})\, e^{jw\cdot 0}\, dw$

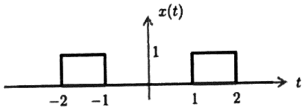
$= 2\pi\, x[0] = 8\pi$

h. $\int_{-\pi}^{\pi} |X(e^{jw})|^2 dw = 2\pi \sum_n |x[n]|^2$

$= 2\pi\, [16+1+9+4+9+1+16]$

$= 112\pi$

3. a) (15 points) Determine the Fourier transform of the continuous-time signal $x(t)$ depicted below:



b) (5 points) Sketch the phase plot for $X(j\omega)$.

(a). $x(t) = g(t) * h(t)$, where $g(t) = \begin{cases} 1 & |t|<\frac{1}{2} \\ 0 & \text{o.w.} \end{cases}$, $h(t)=\delta(t-\frac{3}{2})+\delta(t+\frac{3}{2})$

$G(j\omega)=\int_{-\infty}^{\infty} g(t)e^{-j\omega t}\,dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t}\,dt = \frac{-1}{j\omega}e^{-j\omega t}\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{2\sin\frac{w}{2}}{w}$

$H(j\omega)=\int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt = e^{-j\omega\frac{3}{2}}+e^{j\omega\frac{3}{2}} = 2\cos\frac{3}{2}w$

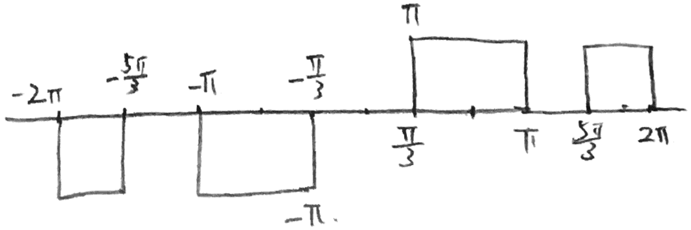
$X(j\omega)=G(j\omega)H(j\omega)=\frac{4\sin\frac{1}{2}w\cos\frac{3}{2}w}{\omega}$

(b). $X(j\omega)=|X(j\omega)|e^{\angle X(j\omega)}$

accounts for sign change

$$\angle X(j\omega)=\begin{cases} 0, & |w|<\frac{\pi}{3} \\ \pm\pi, & \frac{\pi}{3}<|w|<\pi \\ 0, & \pi<|w|<\frac{5\pi}{3} \\ \pm\pi, & \frac{5\pi}{3}<|w|<2\pi \end{cases}$$

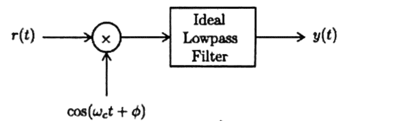
phase is odd. so plot is



3. (20 points) Let $s(t)$ be a real-valued signal for which $S(j\omega)=0$ when $|\omega|>\omega_c$. Amplitude modulation is performed to produce the signal:

$$r(t)=s(t)\cos(\omega_c t)$$

and the demodulation scheme below is applied to $r(t)$ at the receiver. The constant ϕ represents a phase error that arises when the modulator and demodulator are not synchronized. Determine $y(t)$ assuming that the ideal lowpass filter has a cutoff frequency of ω_c and a passband gain of 2. Your answer should depend only on $s(t)$ and ϕ.



Since $r(t)=s(t)\cos(w_c t)$

$R(jw)=\frac{1}{2\pi}S(jw)*\mathcal{F}\{\cos(w_c t)\}$

$=\frac{1}{2\pi}S(jw)*\pi(\delta(w+w_c)+\delta(w-w_c))$

$=\frac{1}{2}S(j(w+w_c))+\frac{1}{2}S(j(w-w_c))$

Let $z(t)=r(t)\cos(w_c t+\phi)$

$Z(jw)=\frac{1}{2\pi}R(jw)*\mathcal{F}\{\cos(w_c t+\phi)\}$

$=\frac{1}{2\pi}R(jw)*\pi(e^{-j\phi}\delta(w+w_c)+e^{j\phi}\delta(w-w_c))$

$=\frac{1}{2}e^{-j\phi}R(j(w+w_c))+\frac{1}{2}e^{j\phi}R(j(w-w_c))$

$=\frac{1}{4}e^{-j\phi}[S(j(w+2w_c))+S(jw)]+$

$\frac{1}{4}e^{j\phi}[S(j(w-2w_c))+S(jw)]$

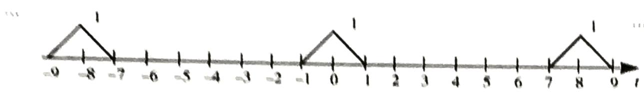
After the LPF, we will only have:

$Y(jw)=\frac{1}{2}e^{-j\phi}S(jw)+\frac{1}{2}e^{j\phi}S(jw)=\cos\phi\, S(jw)$

$y(t)=\mathcal{F}^{-1}\{Y(jw)\}=\cos\phi\, s(t)$ where $\cos\phi$ is the attenuation factor.

Problem 3 (35 points) Fourier Series

[10 pts.] a) $x(t)$ is a periodic function as shown:



$x(t)$ can be represented as a Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$.

Find ω_0 and X_k.

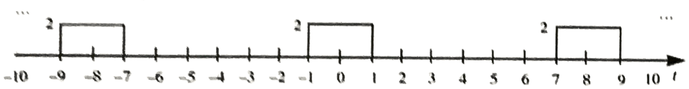
$$\boxed{\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{8} = \frac{\pi}{4}}$$

$$\boxed{X_k = \frac{8(\sin\frac{k\pi}{8})^2}{(k\pi)^2}}$$

Since we know that $x(t)$ is convolution of $h(t) = \sum_{n=-\infty}^{\infty} \Pi(t-8n)$ w/ itself, and we know $b_k = \frac{1}{T_0}\int h(t) e^{-j\frac{k2\pi}{T_0}t}\,dt = \frac{\sin k\frac{\pi}{T_0}}{k\pi}$

we know $X_k = T_0 b_k^2 = \frac{8\sin(\frac{k\pi}{8})^2}{(k\pi)^2}$

[10 pts.] b) $a(t)$ is a periodic function as shown:



$a(t)$ can be represented as a Fourier Series $(t) = \sum_{k=-\infty}^{\infty} \frac{2\sin(k\pi/4)}{k\pi} e^{j\frac{k\pi}{4}t}$.

What is the time average power in $a(t)$? $\underline{\quad 1 \quad}$

Let $b_k = \frac{2\sin(k\pi/4)}{k\pi}$, $b_0 = \frac{1}{2}$

What is the time average power at the fundamental frequency in $a(t)$? $\underline{\quad 4/\pi^2 \quad}$

$|b_1| = |b_{-1}| = \left|\frac{2\sin\frac{\pi}{4}}{\pi}\right| = \frac{\sqrt{2}}{\pi}$, the power is $\frac{4}{\pi^2}$

Problem 3 (cont.)

[~~15~~ pts.] c) Consider a system whose behavior is specified by the differential equation

10

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{d^2x(t)}{dt^2} + \frac{\pi^2}{16}x(t)$$

with input $x(t)$ and output $y(t)$.

If the input to the system is the periodic function $a(t)$ from part ~~3c~~ 3b above, express the output as a Fourier Series $b(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\frac{k\pi}{4}t}$. Find b_k.

$$\boxed{b_k = \frac{\pi}{8} \frac{(1-k^2)}{(j\frac{k\pi}{4}+1)^2} \frac{\sin\frac{k\pi}{4}}{k}}$$

Bonus (2 pts.) (only applicable if you got b_k right): What is the time average power at the fundamental frequency in $b(t)$? $\underline{\quad 0 \quad}$

(c). From PS#2-prob5, we see that $e^{j\omega t}$ is an eigenfunction input, and let the corresponding output be $\tilde{y}(t) = He^{j\omega t}$, we have

$$-H\omega^2 e^{j\omega t} + 2j\omega H e^{j\omega t} + He^{j\omega t} = -\omega^2 e^{j\omega t} + \frac{\pi^2}{16}e^{j\omega t}$$

$$\Rightarrow H(j\omega) = \frac{\omega^2 - \frac{\pi^2}{16}}{\omega^2 - 2j\omega - 1} = \frac{(\omega+\frac{\pi}{4})(\omega-\frac{\pi}{4})}{-(j\omega+1)^2}$$

Since $x(t)$ is weighted sum of eigenfunctions $e^{j\frac{k\pi}{4}t}$

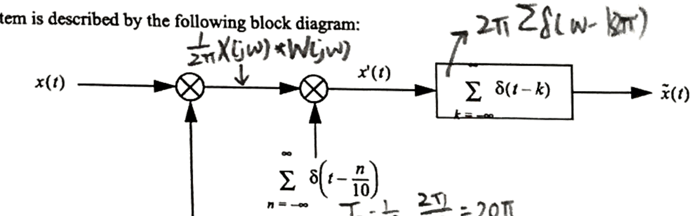
$$y(t) = \sum_{k=-\infty}^{\infty} H(j\tfrac{k\pi}{4}) \frac{2\sin(k\pi/4)}{k\pi} e^{j\frac{k\pi}{4}t}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\frac{\frac{\pi^2}{16}(1-k^2)}{(j\frac{k\pi}{4}+1)^2} \frac{2\sin\frac{k\pi}{4}}{k\pi}}_{b_k} e^{j\frac{k\pi}{4}t}$$

Bonus: $b_1 = b_{-1} = 0$

Problem 7 (35 points)

A system is described by the following block diagram:

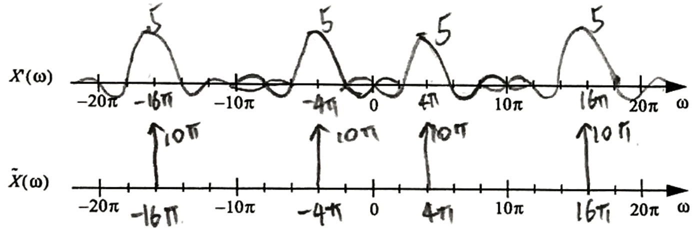


$T_s=\frac{1}{10},\ \frac{2\pi}{T_s}=20\pi$

$$w(t)=\begin{cases}1 & \text{if } -.5\le t<.5\\ 0 & \text{otherwise}\end{cases}$$

$W(jw)=\frac{2\sin\frac{w}{2}}{w}$

[30 pts.] a) Let $x(t)=\cos 4\pi t$. Sketch $X'(\omega)$ and $\tilde{X}(\omega)$, labelling peak magnitude, zero crossing(s), and spacing. (Hint: $X'(\omega)$ and $\tilde{X}(\omega)$ should be real.)



$$\mathcal{F}\{x(t)\cdot w(t)\}=\frac{1}{2\pi}X(jw)*W(jw)=\frac{1}{2\pi}\left[\pi(\delta(w-4\pi)+\delta(w+4\pi))\right]*W(jw)$$

$$=\frac{1}{2}W(j(w-4\pi))+\frac{1}{2}W(j(w+4\pi))$$

where $W(jw)$ is sinc w/ zero crossing at 2π multiples, except for $w=0$ which has value of 1.

$$\mathcal{F}\{x'(t)\}=\mathcal{F}\{(x(t)\cdot w(t))\cdot\sum\delta(t-\tfrac{n}{10})\}=\frac{1}{2\pi}\mathcal{F}\{x(t)w(t)\}*\frac{2\pi}{\frac{1}{10}}\sum\delta(w-k20\pi)$$

$$=5\sum_{k=\infty}^{\infty}\left[W(j(w-4\pi-k20\pi))+W(j(w+4\pi-k20\pi))\right]$$

$\tilde{X}(jw)$ are simply samples of $X'(jw)$ at 2π multiples.

[5 pts.] b) What is the relationship between $\tilde{X}(\omega)$ and the 10 point DFT of $x[n]=X[k]$ (where $x[n]=x(0)\ldots x(.9)$)? Explain why. (What is the effect of not shifting the window $w(t)$ by $T/2$?)

The new window $w'(t)=w(t-\frac{1}{2})$, so $\mathcal{F}\{w'(t)\}=W(jw)e^{-jw\frac{1}{2}}$

Using the procedure as above, this window corresponds to FT:

$$\bar{X}'(jw)=5\sum_{k=-\infty}^{\infty}\left[W'(j(w-4\pi-k20\pi))+W'(j(w+4\pi-k20\pi))\right]$$

$$=5\sum_{k=-\infty}^{\infty}\left[W(j(w-4\pi-k20\pi))e^{-j\frac{1}{2}w}+W(j(w+4\pi-k20\pi))e^{-j\frac{1}{2}w}\right]$$

$$\left(\text{Since } e^{-j\frac{1}{2}(w-4\pi-k20\pi)}=e^{-j\frac{1}{2}w}\underbrace{e^{j2\pi}e^{jk10\pi}}_{1}=e^{-j\frac{1}{2}w}\right)$$

$$=5e^{-j\frac{1}{2}w}X'(jw)$$

From the DFT H.O., we know that

$$X[k]=\frac{T_0}{2\pi}\text{area}\left[\bar{X}'(jk\tfrac{2\pi}{T_0})\right],\ w/\ T_0=1,$$

and $\bar{X}'(jk\frac{2\pi}{T_0})=5e^{-j\frac{1}{2}k\frac{2\pi}{T_0}}X'(jk\frac{2\pi}{T_0})$

$$=5e^{-jk\pi}X'(jk2\pi)$$

where $\bar{X}'(jw)$ is found above

Therefore $X[k]=\frac{5e^{-jk\pi}}{2\pi}\text{area}[X'(jk2\pi)]$

However, since $X'(jk2\pi)$ is non-zero only when $k=\pm2,\pm8$, at which point $e^{-jk\pi}=1$, therefore, in this particular case $X[k]=\frac{5}{2\pi}\text{area}[X'(jk2\pi)]$, the shifting window has no effect on DFT.