Professor Fearing
EECS120/Problem Set 2 v 1.01
Fall 2016
Due at 4 pm, Fri. Sep. 9 in HW box under stairs (1st floor Cory) Reading: O\&W Ch 1, Ch2.

Note: $\Pi(t)=u\left(t+\frac{1}{2}\right)-u\left(t-\frac{1}{2}\right)$, and $r(t)=t u(t)$ where $u(t)$ is the unit step.

## 1. (10 pts) What are signals and systems? (Lec 1)

For each example below, identify the input $x$, system $H\}$, and output $y$. Which of $(x, H, y)$ are known a priori, which would need to be calculated or designed? Example: NASDAQ market manipulation. Answer: Input: buy and sell orders. System: security dealer network plus investor behavior plus stock order book, e.g. limit orders. Output: share price. System known, output directly measured, design input to control price.
a) atomic force microscope
b) a heart pacemaker
c) a prosthetic arm reaching for a cup
d) a voice synthesizer
a) input: force between probe and sample
system: laser beam, photon detector
output: imaging of a surface
b) input: electrical activity of the heart
system: computer, doctor analysis
output: electrical impulse
c) input: user input, toggling
system: arm hardware/software
output: movement of arm from previous position
d) input: text from user
system: dictionary of phonemes, computer
output: audio reading of text
2. ( 12 pts ) algebra review

For $f(t)=[u(t+1)-r(t+1)+r(t-1)] u(3-t)$, sketch:
a) $f(t) \quad$ b) $f\left(\frac{t}{2}\right)$
c) $f\left(\frac{3}{2} t+1\right) \quad$ d) $f\left(-\frac{3}{2} t+1\right)$
e) $f(t)-f(-t)$ f) $f(t)+f(-t)$

In each graph, red/green represents the respective components of $f$, blue is the graph of $f$
a) $u(t+1)$ "starts" at $t=-1$
$r(t+1)=(t+1) u(t+1)$ also "starts" at $t=-1$
$r(t-1)=(t-1) u(t-1)$ also "starts" at $t=1$
$u(3-t)$ "ends" at $t=3$

b) $f\left(\frac{t}{2}\right)$ "starts" at $t=-2$
$f\left(\frac{t}{2}\right)$ "ends" at $t=6$

c) $f\left(\frac{3 t}{2}+1\right)$ "starts" at $t=-\frac{4}{3}$ $f\left(\frac{3 t}{2}+1\right)$ "ends" at $t=\frac{4}{3}$

d) $f\left(-\frac{3 t}{2}+1\right)$ "starts" at $t=\frac{4}{3}$ $f\left(-\frac{3 t}{2}+1\right)$ "ends" at $t=-\frac{4}{3}$


Graph below for both e) and f):

$f(t)$ from a), $f(-t)$ : "starts" at $t=1$ $f(-t)$ "ends" at $t=-3$
e)


3. (20 pts) DT and CT Convolution (Lec 2, OW 2.1, 2.2, Arcak Lec 1) Consider a DT or CT linear time invariant operator $H\}$.

Assume that in discrete time $H\{\delta[n]\}=h[n]$. In continuous time, $H\{\delta(t)\}=h(t)$.
Given $x[n]=\delta[n]+\frac{1}{2} \delta[n-1]+\frac{1}{4} \delta[n-2]$, let $y[n]=H\{x[n]\}$.
Also, $x(t)=\delta(t)+\frac{1}{2} \delta(t-1)+\frac{1}{4} \delta(t-2)$, and let $y(t)=H\{x(t)\}$.
a. Using LTI properties directly of $H\{x[n]\}$, find $y[n]$ in terms of $h[n]$.
b. Explicitly using $x_{1}[n]=\sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$, and LTI properties, find $H\left\{x_{1}[n]\right\}$.
c. Using LTI properties directly of $H\{x(t)\}$, find $y(t)$ in terms of $h(t)$.
d. Explicitly using $x_{1}(t)=\int_{\tau=-\infty}^{\infty} x(\tau) \delta(t-\tau) d \tau$ and LTI properties, find $y_{1}(t)=H\left\{x_{1}(t)\right\}$.
e. Repeat part d. above, but now use $h(t)=e^{-t} u(t)$, and sketch $y_{1}(t)$.
a) $y[n]=H\{x[n]\}$
$y[n]=H\left\{\delta[n]+\frac{1}{2} \delta[n-1]+\frac{1}{4} \delta[n-2]\right\}$
Because LTI systems are linear, we can rewrite $y[n]$ as:
$y[n]=H\{\delta[n]\}+\frac{1}{2} H\{\delta[n-1]\}+\frac{1}{4} H\{\delta[n-2]\}$
From the definition given in the problem along with the time invariance property of LTI systems:
$y[n]=h[n]+\frac{1}{2} h[n-1]+\frac{1}{4} h[n-2]$
b)
$y_{1}[n]=H\left\{x_{1}[n]\right\}$
$y_{1}[n]=H\left\{\sum_{k=-\infty}^{\infty} x[k] \delta[n-k]\right\}$
$y_{1}[n]=H\{\ldots+x[-1] \delta[n+1]+x[0] \delta[n-0]+x[1] \delta[n-1]+\ldots\}$
We can apply the linear property:
$y_{1}[n]=\ldots+H\{x[-1] \delta[n+1]\}+H\{x[0] \delta[n-0]\}+H\{x[1] \delta[n-1]\}+\ldots$
Each $x[k]$ is a constant, so we can move them outside of the system:
$y_{1}[n]=\ldots+x[-1] H\{\delta[n+1]\}+x[0] H\{\delta[n-0]\}+x[1] H\{\delta[n-1]+\ldots$
Applying time invariance:
$y_{1}[n]=\ldots+x[-1] h[n+1]+x[0] h[n-0]+x[1] h[n-1]+\ldots$

$$
\begin{aligned}
& y_{1}[n]=\sum_{k=-\infty}^{\infty} x[k] h[n-k] \\
& y_{1}[n]=\sum_{k=-\infty}^{\infty}\left(\delta[k]+\frac{1}{2} \delta[k-1]+\frac{1}{4} \delta[k-2]\right) h[n-k] \\
& y_{1}[n]=h[n]+\frac{1}{2} h[n-1]+\frac{1}{4} h[n-2]
\end{aligned}
$$

c)

$$
\begin{aligned}
& y(t)=H\{x(t)\} \\
& y(t)=H\left\{\delta(t)+\frac{1}{2} \delta(t-1)+\frac{1}{4} \delta(t-2)\right\}
\end{aligned}
$$

Applying linear property of LTI systems:
$y(t)=H\{\delta(t)\}+\frac{1}{2} H\{\delta(t-1)\}+\frac{1}{4} H\{\delta(t-2)\}$
Applying time invariance property of LTI systems:
$y(t)=h(t)+\frac{1}{2} h(t-1)+\frac{1}{4} h(t-2)$
d)
$y_{1}(t)=H\left\{x_{1}(t)\right\}$
$y_{1}(t)=H\left\{\int_{\tau=-\infty}^{\infty} x(\tau) \delta(t-\tau) d \tau\right\}$
$y_{1}(t)=\int_{\tau=-\infty}^{\infty} x(\tau) H\{\delta(t-\tau) d \tau\}$
Applying the time invariance property of LTI systems:
$y_{1}(t)=\int_{\tau=-\infty}^{\infty} x(\tau) h(t-\tau) d \tau$
$y_{1}(t)=\int_{\tau=-\infty}^{\infty}\left(\delta(\tau)+\frac{1}{2} \delta(\tau-1)+\frac{1}{4} \delta(\tau-2)\right) h(t-\tau) d \tau$
$y_{1}(t)=\int_{\tau=-\infty}^{\infty}\left(\delta(\tau)+\frac{1}{2} \delta(\tau-1)+\frac{1}{4} \delta(\tau-2)\right) h(t-\tau) d \tau$
$y_{1}(t)=h(t)+\frac{1}{2} h(t-1)+\frac{1}{4} h(t-2)$
e) Given that $h(t)=e^{-t} u(t)$,
$y_{1}(t)=e^{-t} u(t)+\frac{1}{2} e^{-(t-1)} u(t-1)+\frac{1}{4} e^{-(t-2)} u(t-2)$


## 4. (18 pts) BIBO Stability (Lec 2, OW 2.3, Arcak Lec 1)

For each of the following impulse responses, determine whether the system is BIBO stable. If the system is not BIBO stable, find a bounded input $x(t)$ or $x[n]$ which gives an unbounded output, and show that the output is unbounded for this input. (Here $u(t)$ is unit step.)
a. $h(t)=\sum_{n=-\infty}^{\infty} \Pi(t-6 n)$
a) $h(t)=\sum_{n=-\infty}^{\infty} u\left(t-6 n+\frac{1}{2}\right)-u\left(t-6 n-\frac{1}{2}\right)$
b. $h(t)=u(t-1)$
c. $h(t)=t^{-1} u(t-1)$
d. $h[n]=e^{-n} u[n+4]$
e. $h(t)=\sin (2 \pi t) u(t)$
f. $h[n]=\sum_{k=-\infty}^{\infty} \delta[n-2 k]$

To test for stability, we first check that:
$\int_{t=-\infty}^{\infty}|h(t)| d t<\infty$
$=\int_{t=-\infty}^{\infty}\left|\sum_{n=-\infty}^{\infty} u\left(t-6 n+\frac{1}{2}\right)-u\left(t-6 n-\frac{1}{2}\right)\right| d t$
$=\sum_{n=-\infty}^{\infty} \int_{t=-\infty}^{\infty} u\left(t-6 n+\frac{1}{2}\right)-u\left(t-6 n-\frac{1}{2}\right) d t$
$=\sum_{n=-\infty}^{\infty} \int_{t=-\infty}^{\infty} 1 d t \rightarrow \infty$
The system is not B.I.B.O. stable.
For example, with an input $x(t)=1$, which is bounded by $B=1$ we can see that:
$\int_{\tau=-\infty}^{\infty}\left|x(\tau) \sum_{n=-\infty}^{\infty} u\left(t-\tau-6 n+\frac{1}{2}\right)-u\left(t-\tau-6 n-\frac{1}{2}\right)\right| d \tau$
$\int_{\tau=-\infty}^{\infty}\left|\sum_{n=-\infty}^{\infty} u\left(t-\tau-6 n+\frac{1}{2}\right)-u\left(t-\tau-6 n-\frac{1}{2}\right)\right| d \tau \rightarrow \infty$
b) $h(t)=u(t-1)$

To test for stability, we first check that:
$\int_{t=-\infty}^{\infty}|h(t)| d t<\infty$
$=\int_{t=-\infty}^{\infty}|u(t-1)| d t$
$=\int_{t=1}^{\infty} 1 d t \rightarrow \infty$
The system is not B.I.B.O. stable.
For example, with an input $x(t)=1$, which is bounded by $B=1$ we can see that:
$\int_{\tau=-\infty}^{\infty} x(\tau) u(t-\tau-1) d \tau$
$\int_{\tau=-\infty}^{\infty} u(t-\tau-1) d \tau \rightarrow \infty$
c) $h(t)=\frac{1}{t} u(t-1)$

To test for stability, we first check that:
$\int_{t=-\infty}^{\infty}|h(t)| d t<\infty$
$=\int_{t=-\infty}^{\infty}\left|\frac{1}{t} u(t-1)\right| d t$
$=\int_{t=1}^{\infty} \frac{1}{t} d t$
$=\ln (t)]_{t=1}^{\infty} \rightarrow \infty$
The system is not B.I.B.O. stable.
For example, with an input $x(t)=1$, which is bounded by $B=1$ we can see that:
$\int_{\tau=-\infty}^{\infty} x(\tau) \frac{1}{t-\tau} u(t-\tau-1) d \tau$
$\int_{\tau=-\infty}^{\infty} \frac{1}{t-\tau} u(t-\tau-1) d \tau>\infty$
d) $h[n]=e^{-n} u[n+4]$

To test for stability, we first check that:
$\sum_{n=-\infty}^{\infty}|h[n]|<\infty$
$=\sum_{n=-\infty}^{\infty}\left|e^{-n} u[n+4]\right|$
$=\sum_{n=-4 y}^{\infty} e^{-n}$
Applying the infinte geometric sum formula, since $e^{-1}<1$
$=\frac{e^{4}}{1-e^{-1}}<\infty$
The system is B.I.B.O. stable
e) $h(t)=\sin (2 \pi t) u(t)$

To test for stability, we first check that:
$\int_{t=-\infty}^{\infty}|h(t)| d t<\infty$
$=\int_{t=-\infty}^{\infty}|\sin (2 \pi t) u(t)| d t$
$=\int_{t=0}^{\infty}|\sin (2 \pi t)| d t \rightarrow \infty$
The system is not B.I.B.O. stable
For example, with an input $x(t)=\sin (2 \pi t)$, which is bounded by $B=1$ we can see that:
$\int_{\tau=-\infty}^{\infty} x(\tau) \sin (2 \pi(t-\tau)) u(t-\tau) d \tau$
$\int_{\tau=-\infty}^{\infty} \sin (2 \pi \tau) u(\tau) \sin (2 \pi(t-\tau)) u(t-\tau) d \tau$
$\int_{\tau=-\infty}^{\infty} \sin (2 \pi \tau) \sin (2 \pi(t-\tau)) u(t-\tau) d \tau$
$\int_{\tau=-\infty}^{\infty} \sin (2 \pi \tau) \sin (2 \pi(t-\tau)) u(t-\tau) d \tau$
$\int_{\tau=-\infty}^{\infty}\left(\cos (2 \pi \tau) \sin (2 \pi t) \sin (2 \pi \tau)-\sin ^{2}(2 \pi \tau) \cos (2 \pi t)\right) u(t-\tau) d \tau \rightarrow \infty$
due to the $\sin ^{2}$ term
f) $h[n]=\sum_{k=-\infty}^{\infty} \delta[n-2 k]$

To test for stability, we first check that:
$\sum_{n=-\infty}^{\infty}|h[n]|<\infty$
$=\sum_{n=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty} \delta[n-2 k]\right|$
$=\sum_{n=-\infty}^{\infty} \cdots+\delta[n+2]+\delta[n]+\delta[n-2]+\ldots \rightarrow \infty$
The system is not B.I.B.O. stable
For example, with an input $x[n]=1$, which is bounded by $B=1$ we can see that:
$\sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} \delta[n-k-2 m]$
$=\sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta[n-k-2 m]$
$=\sum_{k=-\infty}^{\infty} \cdots+\delta[n-k+2]+\delta[n-k]+\delta[n-k-2]+\ldots \rightarrow \infty$

## 5. (20 pts) LDE CT and Block Diag (Lec 3, OW 3.2, Arcak Lec 2)

An LTI system with input $x(t)$ and output $y(t)$ is described by the LDE:

$$
\begin{equation*}
\frac{d^{2} y(t)}{d t^{2}}+2 \frac{d y(t)}{d t}+101 y(t)=100 \frac{d x(t)}{d t} \tag{1}
\end{equation*}
$$

a. Draw a block diagram realization of this system using multiply by constant, summation blocks, and the minimum number of integration blocks.
b. What is the steady state response of the system (i.e. the response after all transients have died away) described by the LDE in eq.(1) to a complex exponential input $x(t)=e^{j \omega t}$ ? (Hint: $e^{j \omega t}$ is an eigenfunction, and thus you can assume $y(t)=H e^{j \omega t}$ where $H$ is a complex constant for a given $\omega$.)
c. What are the eigenvalues (determined from $H$ )? Is the system B.I.B.O. stable?
a)


First we get the equation in the form:

$$
\frac{d^{2} y(t)}{d t^{2}}=100 \frac{d x(t)}{d t}-2 \frac{d y(t)}{d t}-101 y(t)
$$

and then integrate both sides:
$\frac{d y(t)}{d t}=100 x(t)-2 y(t)-101 \int y(t)$
b) $x(t)=e^{j \omega t}$
$y(t)=H x(t)=H e^{j \omega t}$
$-\omega^{2} H e^{j \omega t}+2 j \omega H e^{j \omega t}+101 H e^{j \omega t}=100 j \omega e^{j \omega t}$
$H=\frac{-100 j \omega}{\omega^{2}-2 j \omega-101}$
c)

To find the eigenvalues, we set $y=e^{\lambda t}$ and $x=0$.
From there, we can get the characteristic equation:
$\lambda^{2} e^{\lambda t}+2 \lambda e^{\lambda t}+101 e^{\lambda t}=0$
Solving for this equation:
$\lambda=\frac{-2+\sqrt{-804}}{2}, \frac{2-\sqrt{-804}}{2}$
$\lambda=-1+j \sqrt{201},-1-j \sqrt{201}$
Both have real components which are less than zero, so the system is B.I.B.O. stable.

## 6. (20 pts) LDE DT and Block Diag (Lec 3, OW 3.2, Arcak Lec 2)

An LTI system with input $u[n]$ (here $u[]$ is not the unit step) and output $y[n]$ is described by the LDE:

$$
\begin{equation*}
y[n]-\frac{9}{16} y[n-2]=u[n]+10 u[n-1]+25 u[n-2] \tag{2}
\end{equation*}
$$

a. Draw a block diagram realization of this system using multiply by constant, summation blocks, and the minimum number of delay blocks.
b. What is the steady state response of the system (i.e. the response after all transients have died away) described by the LDE in eq. (2) to a complex exponential input $u[n]=e^{j \omega n}$ ? (Hint: $e^{j \omega n}$ is an eigenfunction and thus you can assume $y[n]=H e^{j \omega n}$ where $H$ is a complex constant for a given $\omega$.)
c. Re-write the linear difference equation in state space form:
$\mathbf{x}[n+1]=A \mathbf{x}[n]+B \mathbf{u}[n] \underline{B u[n]}$ and output equation $y[n]=C \mathbf{x}[n]+D u[n]$, specifying the 4 matrices $A, B, C, D$. Note that $A$ is $3 \mathrm{x} 3, \mathrm{x}$ is $3 \mathrm{x} 1, u$ is scalar, $C$ is 1 x 3 , and $D$ is 1 x 1 .
Note $\left.u[n]=\left[\begin{array}{ll}u\left[\begin{array}{ll}n & 2\end{array}\right] \quad u\left[\begin{array}{ll}n & 1\end{array}\right] u[n\end{array}\right]\right]^{T}$.
Also, use $A$ of the form:

$$
A=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
? & ? & ?
\end{array}\right]
$$

where the botton row needs to be determined. (This is controller canonical form).

a)
b) $u[n]=e^{j \omega n}$
$y[n]=H e^{j \omega n}$
$H e^{j \omega n}-\frac{9}{16} H e^{j \omega n}=e^{j \omega n}+10 e^{j \omega(n-1)}+25 e^{j \omega(n-2)}$
$H\left(e^{j \omega n}-\frac{9}{16} e^{j \omega n}\right)=e^{j \omega n}+10 e^{j \omega(n-1)}+25 e^{j \omega(n-2)}$
$H=\frac{1+10 e^{-j \omega}+25 e^{-2 j \omega}}{1-\frac{9}{16} e^{-2 j \omega}}$
c) Since A has the form $\left[\begin{array}{lll}0 & 1 & 0 \\ 0 & 0 & 1 \\ ? & ? & ?\end{array}\right]$ and $\mathbf{x}[n]$ has the form $\left[x[n-3] x[n-2] x[n-1]^{T}\right.$, we can see that we need to find $\mathbf{x}[n]$ in terms of $x[n-1], x[n-2], x[n-3], u[n]$.

We also know that $\mathbf{y}[n]$ needs to be in terms of $x[n-1], x[n-2], x[n-3], u[n]$.
Based on $A$, we know that $x[n-3]$ is $x[n-2]$ passed through a delay block and that $x[n-2]$ is $x[n-1]$ passed through a delay block. Looking at our block diagram, we can set where the different " $x$ " signals will be:


We can set the topmost red dot to be $x[n]$, and as we go down the delay chain, $x[n-$ 1], $x[n-2]$.

From the block diagram, we can see that $x[n]$ is the sum of the input $u[n]$ and $\frac{9}{16} x[n-1]$.
This means $A=\left[\begin{array}{lll}0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{9}{16} & 0\end{array}\right]$ and $B=\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]$.
From the block diagram, we can see $y[n]$ is the sum of $x[n], x[n-1], x[n-2]$ :
$y[n]=x[n]+10 x[n-1]+25 x[n-2]$
$y[n]=u[n]+\frac{9}{16} x[n-2]+10 x[n-1]+25 x[n-2]$
$y[n]=u[n]+10 x[n-1]+25 \frac{9}{16} x[n-2]$
From this, we can get $C=\left[\begin{array}{lll}0 & 25 \frac{9}{16} & 10\end{array}\right]$ and $D=1$
Note: A 2x2 matrix solution was also accepted and is the minimal solution in this case, but a solution can be obtained from the given constraints above :
$A=\left[\begin{array}{cc}0 & 1 \\ \frac{9}{16} & 0\end{array}\right], B=\left[\begin{array}{l}0 \\ 1\end{array}\right]$, given that $\mathbf{x}=\left[\begin{array}{l}x[n-2] \\ x[n-1]\end{array}\right]$
From there, we can obtain $y[n]=x[n]+10 x[n-1]+25 x[n-2]$
$y[n]=u[n]+\frac{9}{16} x[n-2]+10 x[n-1]+25 x[n-2]$
$y[n]=u[n]+10 x[n-1]+25 \frac{9}{16} x[n-2]$
From this, we can get $C=\left[\begin{array}{ll}25 \frac{9}{16} & 10\end{array}\right]$ and $D=1$

