Professor Fearing EECS120/Problem Set 1 v 1.01 Fall 2016

Due at 4 pm, Fri. Sep. 2 in HW box under stairs (1st floor Cory)

Reading: EE16AB notes. This problem set should be review of material from EE16AB. (Please note, vector notation here is $\mathbf{x} = \vec{x}$, u is a scalar, and A is a matrix.)

1. (18 pts) Complex review.

Given $z = x + jy = re^{j\theta}$. Derive the following relations:

a. $zz^* = r^2$ b. $\frac{z}{z^*} = e^{j2\theta}$

c. $(z_1 z_2)^* = z_1^* z_2^*$ d. $(\frac{z_1}{z_2})^* = \frac{z_1^*}{z_2^*}$

e. Show that $\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})^2 - 1 = \cos 2\omega t$

f. Find all integer $k \in \{0, 1, ...15\}$ for which $\sum_{n=0}^{15} e^{j2\pi nk/16} = 0$.

Solution:

(a) Since $z^* = x - jy = re^{-j\theta}$, we have

$$zz^* = re^{j\theta} \cdot re^{-j\theta} = r^2$$

(b) For division and multiplication, it is handy to use the polar representation:

$$\frac{z}{z^*} = \frac{re^{j\theta}}{re^{-j\theta}} = e^{j2\theta}$$

(c) Let $z_1 = x_1 + jy_1 = r_1 e^{j\theta_1}$, and $z_2 = x_2 + jy_2 = r_2 e^{j\theta_2}$, then we can show that:

$$\begin{aligned}(z_1 z_2)^* &= (r_1 r_2 e^{j(\theta_1+\theta_2)})^* = r_1 r_2 e^{-j(\theta_1+\theta_2)} \\ &= r_1 e^{-j\theta_1} r_2 e^{-j\theta_2} = z_1^* z_2^*\end{aligned}$$

(d) Similar to (c), we can derive that:

$$\begin{aligned}\left(\frac{z_1}{z_2}\right)^* &= \left(\frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}\right)^* = \frac{r_1}{r_2} e^{-j(\theta_1-\theta_2)} \\ &= r_1 e^{-j\theta_1} r_2 e^{j\theta_2} = \frac{z_1^*}{z_2^*}\end{aligned}$$

By the way, according to the Euler's formula, for $z = x + jy = re^{j\theta}$, we have $x = r\cos\theta, y = r\sin\theta$.

(e) We apply the Euler's formula to expand the left hand side (LHS):

$$\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})^2 - 1 = \frac{1}{2}e^{2j\omega t} + \frac{1}{2}e^{-2j\omega t} = \cos 2\omega t$$

(f) If $k = 0$, $e^{j2\pi nk/16} = 1$, and the sum $\sum_{n=0}^{15} e^{j2\pi nk/16}$ is equal to 16, which clearly doesn't satisfy the condition. For $k \neq 0$, the sum of geometric series is given by:

$$\sum_{n=0}^{15} e^{j2\pi nk/16} = \frac{1 - e^{j2\pi k}}{1 - e^{j2\pi k/16}} = 0$$

since $e^{j2\pi k} = 1$ for any integer k. Therefore $k \in \{1, ...15\}$ will satisfy the condition.

2. (24 pts) Phasors and Operational Amplifiers

Consider the circuit in Fig. 2. Use the "golden rules" (ideal) ideal op amp assumptions.

a. Using phasors, determine the transfer function $H = \frac{V_o}{V_i}$.

b. Sketch the magnitude Bode plot (log-log scale) for $H(\omega)$.

c. What filter function is performed (low pass, high pass, etc.)?

Solution: (a) Recap that for any sinusoidal time-varying function $x(t)$, it can be represented in the form $x(t) = \Re[Xe^{j\omega t}]$, where X is a time-independent function called the phasor counterpart of $x(t)$. $x(t)$ is defined in the time domain, while phasor X is defined in the phasor domain.

Step 1: To use the phasors, first denote the phasors of $v_i(t), v_o(t)$ as V_i, V_o.

Step 2: Transform the circuits components to phasor domain, i.e., impedance, using the formula: $Z_R = R$ for resistors, $Z_C = \frac{1}{j\omega C}$ for capacitors, and $Z_L = j\omega L$ for inductors, where R, C, L are the resistance, capacitance, and inductance, and ω is the frequency, as is shown in Fig. 2sol.

Step 3: Use KCL and/or KVL equation in the phasor domain, where we apply the "golden rule" of the ideal op amp to obtain the phasor $V_1 = 0$ for $v_1(t) = v_-(t) = v_+(t) = 0$ (ground). By KCL, and let $R = 100k\Omega$, $C = 2\mu F$, we have:

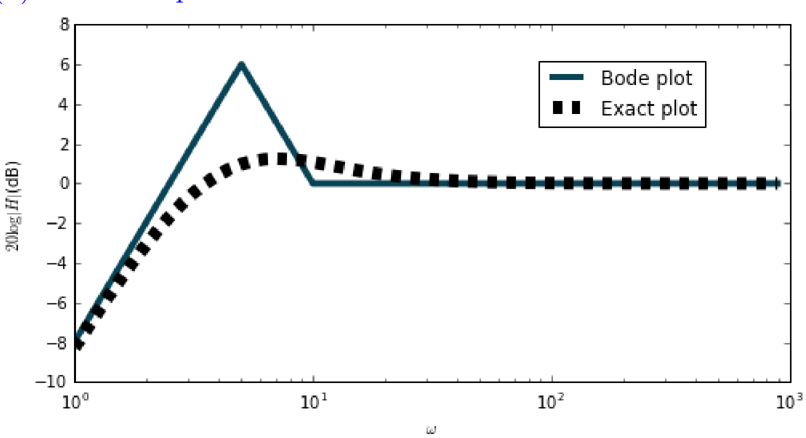
$$\frac{V_i}{R + \frac{1}{j\omega C}} = -\frac{V_o}{R + \frac{R\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}}$$

Step 4: Obtain the transfer function $H = \frac{V_o}{V_i}$:

$$H = \frac{V_o}{V_i} = -\frac{R + \frac{R\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}}{R + \frac{1}{j\omega C}} = -\frac{j\omega CR + \frac{j\omega CR}{j\omega CR + 1}}{j\omega CR + 1} = -\frac{j2\omega CR - \omega^2 C^2 R^2}{(j\omega CR + 1)^2}$$
$$= -\frac{j2\omega/\omega_c + (j\omega/\omega_c)^2}{(j\omega/\omega_c + 1)^2}$$

where $\omega_c = \frac{1}{RC} = 5\text{Hz}$.

(b) The Bode plot is shown below.



(c) The filter is performing high pass.

3. (18 pts) State space

For the circuit shown in Fig. 1, let the state variables x_1 be $v_c(t)$, and x_2 be the inductor current.

The input $u = v_i(t)$ and output $y(t)$ is the voltage across the 1 $M\Omega$ resistor. Let $C = 10^{-6}F$ and $R_2 = 10M\Omega$.
a. Write the differential equation for the circuit in state space form (find A,B,C,D):

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \qquad y = C\mathbf{x} + Du$$

b. Determine the eigenvalues for A. Is the system stable?

Solution: (a) First, using the physical properties of the inductor and capacitors, and let $x_1 = v_c(t)$, $x_2 = i_L(t)$, we have $i_c(t) = C\frac{v_c(t)}{dt} = C\frac{dx_1}{dt}$, and $v_L(t) = L\frac{di_L(t)}{dt} = L\frac{dx_2}{dt}$, where $i_c(t), v_L(t)$ are the current and voltage across the capacitor and inductor, respectively. Now, use the KCL and KVL, and $u = v_i(t)$, we have:

$$u = L\frac{dx_2}{dt} + R_2C\frac{dx_1}{dt} + x_1$$
$$x_2 = C\frac{dx_1}{dt} + \frac{R_2C\frac{dx_1}{dt} + x_1}{R_1}$$

where $R_1 = 1M\Omega$. Rearranging the above equations for $\frac{dx_1}{dt}$ and $\frac{dx_2}{dt}$, we have:

$$\frac{dx_1}{dt} = -\frac{1}{R_1C + R_2C}x_1 + \frac{R_1}{R_1C + R_2C}x_2$$
$$\frac{dx_2}{dt} = -\frac{1}{L}x_1 + \frac{1}{L}u + \frac{R_2}{R_1L + R_2L}x_1 - \frac{R_1R_2}{R_1L + R_2L}x_2$$

Therefore, using $\mathbf{x} = [x_1\ x_2]^T$, we have:

$$\dot{\mathbf{x}} = \begin{bmatrix} -\frac{1}{R_1C+R_2C} & \frac{R_1}{R_1C+R_2C} \\ -\frac{1}{L} + \frac{R_2}{R_1L+R_2L} & -\frac{R_1R_2}{R_1L+R_2L} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}\mathbf{u}$$

Also, we have

$$y(t) = R_2C\frac{dx_1}{dt} + x_1 = -\frac{R_2}{R_1 + R_2}x_1 + \frac{R_1R_2}{R_1 + R_2}x_2 + x_1$$
$$= \begin{bmatrix} \frac{R_1}{R_1+R_2} & \frac{R_1R_2}{R_1+R_2} \end{bmatrix}\mathbf{x}$$

Therefore, we have $A = \begin{bmatrix} -\frac{1}{R_1C+R_2C} & \frac{R_1}{R_1C+R_2C} \\ -\frac{1}{L} + \frac{R_2}{R_1L+R_2L} & -\frac{R_1R_2}{R_1L+R_2L} \end{bmatrix} = \begin{bmatrix} -\frac{1}{11} & \frac{10^6}{11} \\ -\frac{1}{11} & -\frac{10^7}{11} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} \frac{R_1}{R_1+R_2} & \frac{R_1R_2}{R_1+R_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{11} & \frac{10^7}{11} \end{bmatrix}$, and $D = 0$.

(b) To obtain the eigenvalues, we use the characteristic equations:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\frac{1}{11} - \lambda & \frac{10^6}{11} \\ -\frac{1}{11} & -\frac{10^7}{11} - \lambda \end{bmatrix}\right) = \lambda^2 + \frac{1 + 10^7}{11} + \frac{10^6}{11^2}$$

Therefore, the eigenvalues are $\lambda_1 = -0.1$, $\lambda_2 = -909090$. Since both of them are negative values, the system is stable.

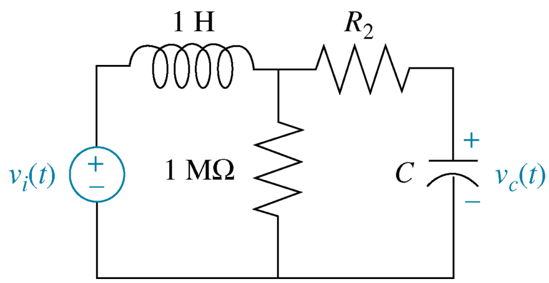


Fig. 1

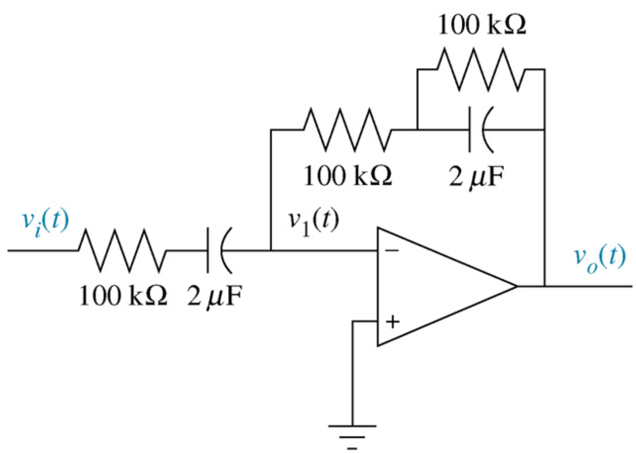


Fig. 2

4. (20 pts) LDE solutions

Consider $\dot{\mathbf{x}} = A\mathbf{x}$ where initial condition $\mathbf{x}_o = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ -10 & -11 \end{bmatrix}$

a. Show that the general solution is $c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$ where λ_1, λ_2 are eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, and find eigenvalues and eigenvectors.
b. Plot each component of the solution and the phase portrait. The phase portrait is a 2 dimensional plot of $x_1(t)$ vs $x_2(t)$. (Hand sketch is fine.)

Solution: (a) To show that $c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$ is the general solution to the LDE, simply substitute the solution to the differential equation:

$$\begin{aligned} LHS &= \frac{d}{dt}\left(c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2\right) = c_1\lambda_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2\lambda_2 e^{\lambda_2 t}\mathbf{v}_2 \\ RHS &= A\left(c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2\right) = c_1 e^{\lambda_1 t}A\mathbf{v}_1 + c_2 e^{\lambda_2 t}A\mathbf{v}_2 \\ &= c_1\lambda_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2\lambda_2 e^{\lambda_2 t}\mathbf{v}_2 = LHS \end{aligned}$$

where we use the fact that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A.

To find the eigenvalues of A, we use the characteristic equation:

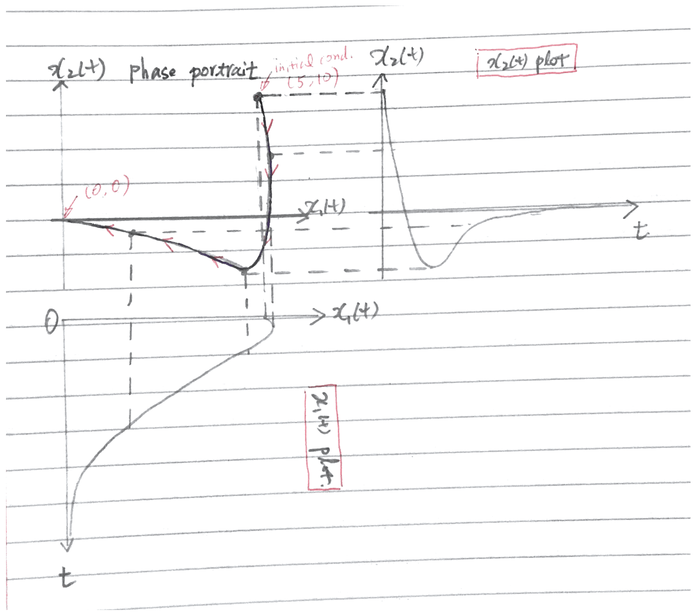
$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -10 & -11-\lambda \end{bmatrix}\right) = \lambda^2 + 11\lambda + 10 = (\lambda+1)(\lambda+10)$$

Therefore the eigenvalues are $\lambda_1 = -1, \lambda_2 = -10$. To find the eigenvectors, let $\mathbf{v_1} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, then we have $A\mathbf{v_1} = \begin{bmatrix} v_2 \\ -10v_1 - 11v_2 \end{bmatrix} = -\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, therefore, $\mathbf{v_1} = [\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}}]^T$. Similarly, let $\mathbf{v_2} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, we have $A\mathbf{v_2} = \begin{bmatrix} w_2 \\ -10w_1 - 11w_2 \end{bmatrix} = -10\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, so $\mathbf{v_2} = [\frac{1}{\sqrt{101}} \quad -\frac{10}{\sqrt{101}}]^T$.

(b) First, let's find the solution to the LDE given the initial condition.

$$\begin{aligned} \frac{1}{\sqrt{2}}c_1 + \frac{1}{\sqrt{101}}c_2 &= 5 \\ -\frac{1}{\sqrt{2}}c_1 - \frac{10}{\sqrt{101}}c_2 &= 10 \end{aligned}$$

where we find $c_1 = \frac{20\sqrt{2}}{3}, c_2 = -\frac{5\sqrt{101}}{3}$. To plot the phase portrait, we start from the initial condition $\mathbf{x}_o = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$ and plot the $x_1(t) = \frac{20}{3}e^{-t} - \frac{5}{3}e^{-10t}$ vs $x_2(t) = -\frac{20}{3}e^{-t} + \frac{50}{3}e^{-10t}$ components. Note that as $t \to \infty$, both components decacy to 0. Here we show one way of ploting it, by first sketch the evolution of $x_1(t)$ (bottom) and $x_2(t)$ (top right), and then find corresponding points on the phase portrait figure.



5. (20 pts) DFT basics

The Discrete Time Fourier Series (DTFS) is defined as

$$a[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{\frac{-j2\pi nk}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]W_N^{nk}$$

where $W_N \equiv e^{-j2\pi/N}$, and $k \in \{0, 1, ..., N-1\}$.

a. Show that the DTFS can be written as $\mathbf{a} = U\mathbf{x}$ for $N = 4$, and write U in terms of W_4.

b. What special property or properties do the columns of U have?

c. For $N = 4$, and $x[n] = \cos\frac{\pi n}{2}$, find $\mathbf{a}$ in terms of W_4 (simplify).

d. For $N = 8$, and $\mathbf{x} = [0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ ]^T$, find $\mathbf{a}$ in terms of W_8 (simplify).

Solution: (a) By the definition of DTFS, $U = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$.

For $\mathbf{a} = [a_0\ a_1\ \cdots\ a_{N-1}]^T$ and $\mathbf{x} = [x_0\ x_1\ \cdots\ x_{N-1}]^T$, we have $\mathbf{a} = U\mathbf{x}$. For $N = 4$, we have:

$$U = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

(b) Since it can be verified that U is orthogonal matrix, the columns of U are orthogonal among each other. The orthogonality can be seen by taking the dot product of column k and p:

$$U_{:,k} \cdot U^{\star}_{:,p} = \frac{1}{N^2}\sum_{n=0}^{N-1} W_N^{nk}W_N^{-np} = 0, \forall k \neq p$$

which follows by the reasoning of prob1 (f).

(c) By Euler's formula, $x[n] = \frac{1}{2}W_4^n + \frac{1}{2}W_4^{-n}$, therefore, $a[k]$ is given by:

$$a[k] = \frac{1}{4}\sum_{n=0}^{3}\left(\frac{1}{2}W_4^n + \frac{1}{2}W_4^{-n}\right)W_4^{nk}$$
$$= \begin{cases} 0 & k \neq 1,3 \\ \frac{1}{2} & k = 1,3 \end{cases}$$

where we use the fact that $W_4^{4n} = 1$. Therefore, $\mathbf{a} = [0\ \frac{1}{2}\ 0\ \frac{1}{2}]$.

(d) By the definition of DTFS:

$$a[k] = \frac{1}{8}\sum_{n=0}^{7} x[n]W_8^{nk} = \frac{1}{8}W_8^{4k} = \frac{1}{8}(-1)^k$$