

## Discussion 3: Graphical Convolution and CT Fourier Series

## 1. Graphical Convolution

Consider Problem 3e from PS2.

Given input,  $x(t) = \delta(t) + \frac{1}{2}\delta(t-1) + \frac{1}{4}\delta(t-2)$ , and impulse response,  $h(t) = e^{-t}u(t)$ , find the output using the convolution integral,

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

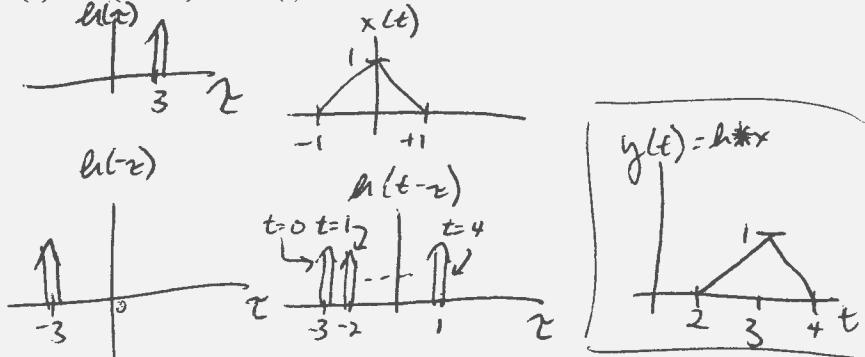
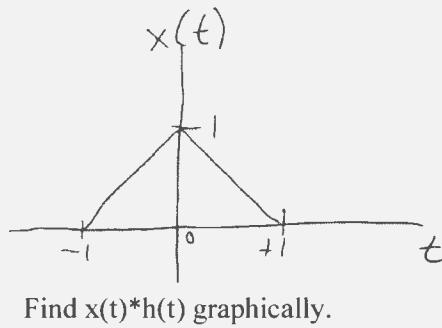
Using the convolution integral, we found that this convolution has the form:

$$y(t) = e^{-t}u(t) + \frac{1}{2}e^{-(t-1)}u(t-1) + \frac{1}{4}e^{-(t-2)}u(t-2)$$

The “flip-and-slide” method of convolution is like watching the input “slide into” the system. If we flip  $x(\tau)$  about  $\tau = 0$  and slide it to the right, we can see that the first impulse “hits”  $h(\tau)$  at  $t = 0$ . At  $t = 1$ , we’ve slid the flipped version of  $x(\tau)$  by 1 unit, and the 2<sup>nd</sup> impulse “hits”  $h(\tau)$ . At  $t = 2$ , we’ve slid the flipped version of  $x(\tau)$  by 2 units, and the 3<sup>rd</sup> impulse “hits”  $h(\tau)$ .

Practice the following graphical convolutions:

- (a) Let  $h(t)$  be a time-shifted delta function:  $h(t) = \delta(t-3)$ . Let  $x(t)$  be a “tent function”:

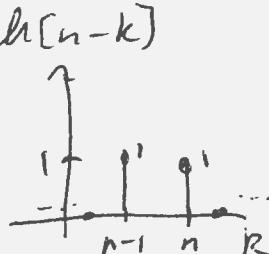
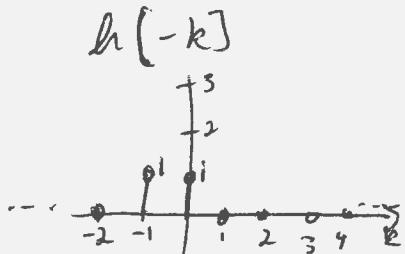
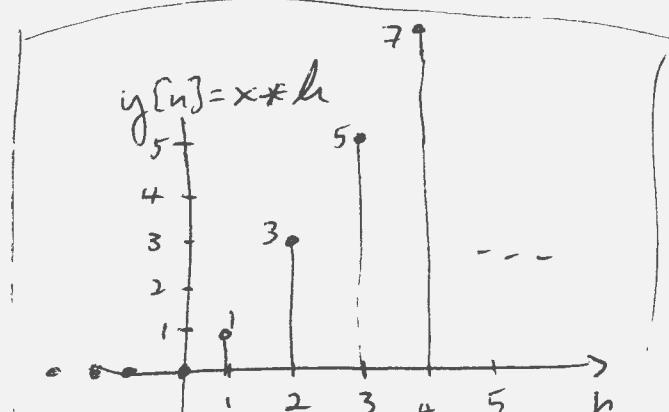
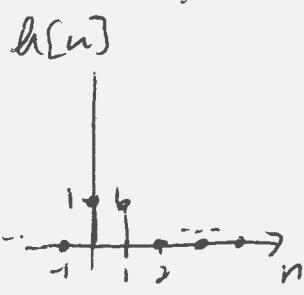
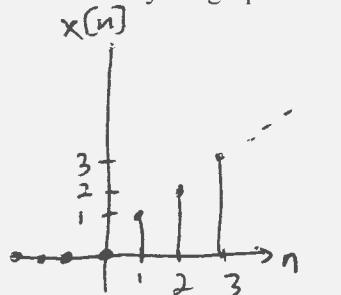


- (b) Let  $x[n]$  be a ramp starting at zero ( $x[n] = n \cdot u[n]$ ) and let  $h[n]$  be two delta functions:

$$h[n] = \delta[n] + \delta[n-1]$$

Find  $x(t)*h(t)$  graphically.

Check your graphical result with the analytical result.



## 2. Continuous Time Fourier Series

The Fourier Series coefficients for a given periodic signal,  $x(t)$ , are given by the analysis equation:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\frac{2\pi}{T}t} dt$$

Where  $T$  is the fundamental period of  $x(t)$ . The signal can be written using the Fourier coefficients in the synthesis equation:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} dt$$

Prove the following properties of CT Fourier Series:

- (a) Linearity: Given  $x(t) \leftrightarrow a_k$  and  $y(t) \leftrightarrow b_k$  both periodic with fundamental period  $T$ ,  
show  $z(t) = Ax(t) + By(t) \leftrightarrow c_k = Aa_k + Bb_k$

$$z(t) = Ax(t) + By(t) = A \sum_k a_k e^{jk\frac{2\pi}{T}t} + B \sum_k b_k e^{jk\frac{2\pi}{T}t} = \sum_k (Aa_k + Bb_k) e^{jk\frac{2\pi}{T}t} = \boxed{\sum_k c_k e^{jk\frac{2\pi}{T}t}} \quad \text{QED}$$

- (b) Time-shifting: Given  $x(t) \leftrightarrow a_k$  with fundamental period  $T$ ,  $y(t) = x(t - t_0)$ ,

$$\text{show } y(t) \leftrightarrow b_k = e^{-jk\frac{2\pi}{T}t_0} a_k$$

$$b_k = \frac{1}{T} \int_{\langle T \rangle} x(t-t_0) e^{-jk\frac{2\pi}{T}t} dt = \frac{1}{T} \int_{\langle T \rangle} x(\tau) e^{-jk\frac{2\pi}{T}(\tau+t_0)} d\tau = \frac{e^{-jk\frac{2\pi}{T}t_0}}{T} \int_{\langle T \rangle} x(\tau) e^{-jk\frac{2\pi}{T}\tau} d\tau$$

$$\text{let } \tau = t - t_0, \Rightarrow t = \tau + t_0$$

$$b_k = \boxed{e^{-jk\frac{2\pi}{T}t_0} a_k}$$

- (c) Time-scaling: Given  $x(t) \leftrightarrow a_k$  with fundamental period  $T$ ,  $y(t) = x(at)$ ,

$$\text{show } y(t) \leftrightarrow b_k = a_k$$

$$y(t) \text{ has fundamental period } \frac{T}{\alpha} \quad \text{Let } \tau = at \quad d\tau = a dt$$

$$b_k = \frac{1}{T/\alpha} \int_{\langle T/\alpha \rangle} x(at) e^{-jk\frac{2\pi}{T/\alpha}t} dt = \frac{\alpha}{T} \int_0^{\frac{T}{\alpha}} x(a\tau) e^{-jk\frac{2\pi}{T}\tau} d\tau = \frac{\alpha}{T} \int_0^T x(\tau) e^{-jk\frac{2\pi}{T}\tau} d\tau$$

$$b_k = \boxed{\frac{1}{T} \int_0^T x(t) e^{-jk\frac{2\pi}{T}t} dt = a_k} \quad \text{QED}$$

- (d) Consider a quarter-period cosine:

$$x_1(t) = \begin{cases} \cos(\omega_0 t) & \frac{2\pi}{\omega_0} \left( n - \frac{1}{8} \right) < t < \frac{2\pi}{\omega_0} \left( n + \frac{1}{8} \right) \\ 0 & \text{otherwise} \end{cases}$$

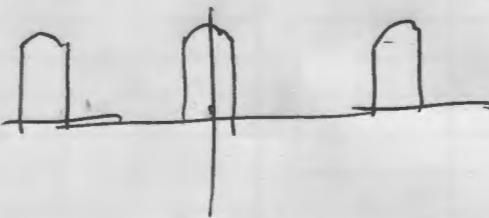
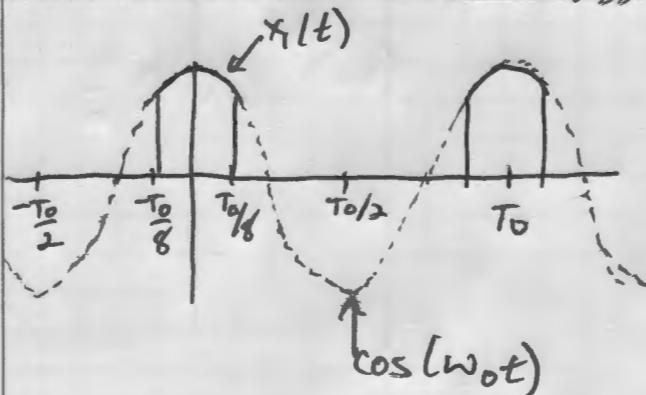
This looks like a “truncated” cosine. Find the Fourier Series coefficients of this signal.

# Problem 2d

EE120

Discussion 3

9/12/2016  
Phil and Ming



$$T_0 = \frac{2\pi}{\omega_0}$$

$$\omega_0 \frac{T_0}{8} = \omega_0 \frac{2\pi}{\omega_0} \frac{1}{8} = \frac{\pi}{4}$$

$$2\omega_0 T_0 = 2 \cdot 2\pi = 4\pi$$

$$a_R = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega_0 t} dt = \frac{1}{T_0} \int_{-T_0/8}^{T_0/8} \cos(\omega_0 t) e^{-j\omega_0 t} dt$$

$$a_R = \frac{1}{T_0} \int_{-T_0/8}^{T_0/8} \frac{1}{2} (e^{+j\omega_0 t} + e^{-j\omega_0 t}) e^{-j\omega_0 t} dt$$

$$a_R = \frac{1}{2T_0} \int_{-T_0/8}^{T_0/8} [e^{j\omega_0(-k)t} + e^{-j\omega_0(1+k)t}] dt$$

$$a_R = \frac{1}{2T_0} \left[ \frac{e^{j\omega_0(-k)t}}{j\omega_0(-k)} + \frac{e^{-j\omega_0(1+k)t}}{-j\omega_0(1+k)} \right] \Big|_{-T_0/8}^{T_0/8}$$

$$a_R = \frac{1}{T_0} \left[ \frac{1}{j\omega_0(1-k)} \left( \frac{e^{j\omega_0(-k)T_0/8}}{2j} - \frac{e^{-j\omega_0(1-k)T_0/8}}{2j} \right) \right. \\ \left. + \frac{1}{j\omega_0(1+k)} \left( \frac{e^{j\omega_0(1+k)T_0/8}}{2j} - \frac{e^{-j\omega_0(1+k)T_0/8}}{2j} \right) \right]$$

$$a_R = \frac{1}{T_0} \left[ \frac{\sin(\omega_0(1-k)\frac{T_0}{8})}{\omega_0(1-k)} + \frac{\sin(\omega_0(1+k)\frac{T_0}{8})}{\omega_0(1+k)} \right]$$

$$a_R = \frac{1}{2\pi} \left( \frac{\sin(\frac{\pi}{4}(1-k))}{1-k} + \frac{\sin(\frac{\pi}{4}(1+k))}{1+k} \right)$$

$$\text{for } k=1, a_1 = \frac{1}{T_0} \int_{-T_0/8}^{T_0/8} (e^0 + e^{-j2\omega_0 t}) dt$$

$$a_{+1} = \frac{1}{2T_0} \left[ t + \frac{e^{-j2\omega_0 t}}{-j2\omega_0} \right] \Big|_{-T_0/8}^{T_0/8}$$

$$a_{+1} = \frac{1}{2T_0} \left[ \frac{T_0}{4} + \frac{e^{-j2\omega_0 \frac{T_0}{8}} - e^{+j2\omega_0 \frac{T_0}{8}}}{-2j\omega_0} \right]$$

$$a_{+1} = \frac{1}{8} + \frac{\sin(\frac{2\pi}{4})}{2\omega_0 T_0} = \frac{1}{8} + \frac{1}{4\pi}$$

Similarly,

$$a_{-1} = \frac{1}{8} + \frac{1}{4\pi}$$