The z-Transform

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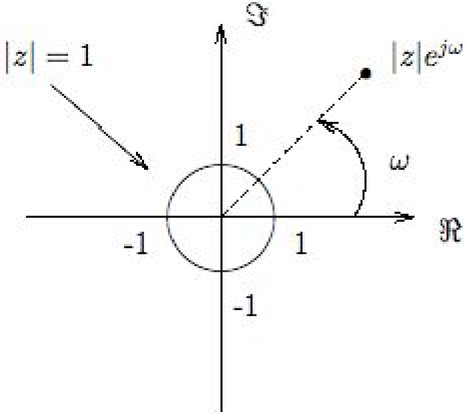
Z transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n},$$

where z is a complex variable, i.e. $z = |z|e^{j\omega}$.



DTFT can be interpreted as the z-transform evaluated on the unit circle,

$$X(e^{j\omega}) = X(z)|_{z=e^{j\omega}}$$

Comparisons between Laplace and Z transform

BASICS OF TWO-SIDED LAPLACE AND Z TRANSFORMS			
DEFINITION	**APPLICATIN**	**IMPORTANCE**	**CONVOLUTION**
$L[x(t)]=\int x(t)e^{-st}dt$	Differential eqns.	$L[dx^N/dt^N]=s^N X(s)$	$L[x(t)*y(t)]=X(s)Y(s)$
$Z[x(n)]=\sum x(n)z^{-n}$	Difference eqns.	$Z[x(n-N)]=z^{-N}X(z)$	$Z[x(n)*y(n)]=X(z)Y(z)$
PROPERTIES OF TWO-SIDED LAPLACE AND Z TRANSFORMS			
TIME SCALE	**REVERSAL**	**INITIAL VALUE**	**TIME MULTIPLY**
$L[e^{at}x(t)]=X(s-a)$	$L[x(-t)]=X(-s)$	$x(0)=\lim_{s\to\infty} sX(s)$	$L[tx(t)]=-dX(s)/ds$
$Z[a^n x(n)]=X(z/a)$	$Z[x(-n)]=X(1/z)$	$x(0)=\lim_{z\to\infty} X(z)$	$Z[nx(n)]=-zdX(z)/dz$
EXAMPLES OF TWO-SIDED LAPLACE AND Z TRANSFORMS			
IMPULSE	**UNIT STEP**	**CAUSAL EXP.**	**ANTICAUSAL EXP.**
$L[\delta(t)]=1$	$L[u(t)]=1/s$	$L[e^{at}u(t)]=1/(s-a)$	$L[-e^{at}u(-t)]=1/(s-a)$
$Z[\delta(n)]=1$	$Z[u(n)]=z/(z-1)$	$Z[a^n u(n)]=z/(z-a)$	$Z[-a^n u(-n-1)]=z/(z-a)$

http://web.eecs.umich.edu/~aey/eecs451/lectures/zl.html

Properties of ROC of Z-Transform

SIGNAL TYPE	CAUSAL	ANTICAUSAL	TWO-SIDED
FINITE LENGTH	$\{x(0) \ldots x(N)\}$	$\{X(-N) \ldots x(0)\}$	$\{x(-M) \ldots x(N)\}$
	$\{z: 0<\lvert z\rvert\leq\infty\}$	$\{z: 0\leq\lvert z\rvert<\infty\}$	$\{z: 0<\lvert z\rvert<\infty\}$
ONE SINGLE EXPONENTIAL	$a^n u(n)$	$-a^n u(-n-1)$	$a^n u(n)+b^n u(-n-1)$
	$\{z: \lvert z\rvert>\lvert a\rvert\}$	$\{z: \lvert z\rvert<\lvert a\rvert\}$	$\{z: \lvert a\rvert<\lvert z\rvert<\lvert b\rvert\}$
SUM SEVERAL EXPONENTIALS	$\Sigma c_i p_i^n u(n)$	$\Sigma d_i q_i^n u(-n-1)$	SUM OF THESE 2
	$\lvert z\rvert>\max[\lvert p_i\rvert]$	$\lvert z\rvert<\min[\lvert q_i\rvert]$	$\max[\lvert p_i\rvert]<\lvert z\rvert<\min[q_i]$

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Causality and Stability

- A discrete time LTI system is causal when
  - ROC is outside the outermost pole.
  - In the transfer function H[z], the order of numerator cannot be greater than the order of denominator.
- A discrete time LTI system is stable when
  - its system function H[z] include unit circle $|z|=1$.
  - all poles of the transfer function lay inside the unit circle $|z|=1$.