Professor Fearing EECS120/Discussion 1 GSI: Ming and Alanna

1. **DFT basics**

We can think of a real-world signal that is a function of time $x(t)$. By recording its values at regular intervals, we can represent it as a vector of discrete samples $\mathbf{x}$, of length N. Let $\mathbf{x} = \begin{bmatrix} x[0] & x[1] & \cdots & x[N-1] \end{bmatrix}^T$, and $\mathbf{X} = \begin{bmatrix} X[0] & \dots & X[N-1] \end{bmatrix}^T$ be the signal $\mathbf{x}$ represented in the frequency domain, that is,

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi nk}{N}} \qquad \text{and} \qquad x[n] = \sum_{k=0}^{N-1} X[k] e^{\frac{j2\pi nk}{N}}$$

(a) For the signal $x[n] = \cos(\frac{2\pi K}{N} n)$, compute the DFT coefficients $\mathbf{X}$, where $\mathbf{x}$ is given by:

$$\mathbf{x} = \begin{bmatrix} \cos(\frac{2\pi K}{N}(0)) & \cos(\frac{2\pi K}{N}(1)) & \cdots & \cos(\frac{2\pi K}{N}(n-1)) \end{bmatrix}^T.$$

In the parts below, please note that the signal $\mathbf{y}$ is also length N, and that its DFT is defined in a manner identical to that of $\mathbf{x}$.

Prove the following properties:

(b) Linearity: $\alpha x[n] + \beta y[n] \longleftrightarrow \alpha X[k] + \beta Y[k]$, where α and β are scalar constants.

(c) Time-Shifting Property: $x[n-M] \longleftrightarrow X[k]\, e^{\frac{-j2\pi Mk}{N}}$, where M is an integer.

(d) Frequency Shifting Property: $e^{\frac{j2\pi nm}{N}} x(n) \longleftrightarrow X[k-m]$, where m is an integer.

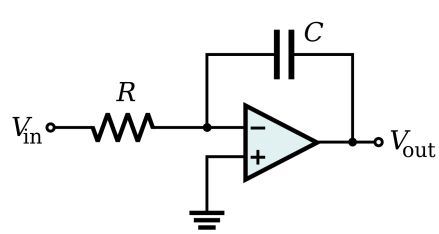
(e) Parseval-Plancherel-Rayleigh Identity

$$\frac{1}{N}\sum_{n=0}^{N-1} x[n]\, y^*[n] = \sum_{k=0}^{N-1} X[k]\, Y^*[k]$$

In the special case where $\mathbf{x} = \mathbf{y}$ the identity above expresses the *energy* of the signal in the frequency domain (i.e., in terms of its spectrum):

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{k=0}^{N-1} |X[k]|^2$$

2. Op-Amps and Transfer Functions



(a) Write the frequency response function $H(\omega) = \frac{V_{out}}{V_{in}}$ for the circuit.

(b) If $R = 1\text{k}\Omega$ and $C = 100\text{nF}$, plot the log-magnitude of $H(\omega)$ and label important magnitudes and frequencies. (Again assume that $H(-\omega) = H(\omega)$.)

(c) What does this circuit do?

Hint: Try to set up the differential equation relating V_{out} and V_{in} to work this out.

3. Matrix differential equations (take-home exercise)

In this problem, we consider ordinary differential equations which can be written in the following form

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} , \tag{1}$$

where x, y are variables depending on t, $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$, and A is a 2×2 matrix with constant coefficients. We call (1) a matrix differential equation.

1. Suppose we have a system of ordinary differential equations

$$x' = 8x + 7y \tag{2}$$
$$y' = -4x - 3y \tag{3}$$

   Write this in the form of (1).

2. Compute the eigenvalues and eigenvectors of the matrix A from the previous part.

3. We claim that the solution for $x(t), y(t)$ is of the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_0 e^{\lambda_0 t} \vec{v}_0 + c_1 e^{\lambda_1 t} \vec{v}_1 ,$$

   where c_0, c_1 are constants, and λ_0, λ_1 are the eigenvalues of A with eigenvectors $\vec{v}_0, \vec{v}_1$ respectively. Suppose that the initial conditions are $x(0) = 1, y(0) = 1$. Solve for the constants c_0, c_1.

4. Verify that the solution for $x(t), y(t)$ found in the previous part satisfies the original system of differential equations (2), (3).

5. We now apply the method above to solve a second order ordinary differential equation. Suppose we have the system

$$z''(t) - 5z'(t) + 6z(t) = 0 \tag{4}$$

   Write this in the form of (1), by using the change of variables $x(t) = z(t)$, $y(t) = z'(t)$.

6. Solve the system in (4) with the initial conditions $z(0) = 1$, $z'(0) = 1$, using the method developed in parts (b) and (c).