Professor Fearing	DTFT from FT v1.0	Fall 2016
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Introduction

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The discrete time Fourier Transform can be found by taking the continuous time (CT) Fourier Transform of a sampled signal. The basic DTFT is mostly straight forward, but there are a few subtle points considered in this handout.

- What are the steps in finding the DTFT using CTFT operations?
- What is the difference between ω in the CTFT and Ω in the DTFT?
- How does the sample period T effect the height of the DTFT?
- In general the DTFT is just the CTFT of the sampled signal multiplied by T, and frequency normalized. Why does this rule not work for FT which contain $\delta(\omega)$?

Deriving discrete time Fourier Transform

Notation: the continuous time Fourier Transform $X(j\omega)$ has frequency $-\infty < \omega < \infty$. The normalized frequency Ω for the DTFT $X(e^{j\Omega})$ has $0 \le \Omega < 2\pi$ (or any period of 2π).

Consider a continuous time signal x(t) with Fourier Transform $X(j\omega)$. The signal is sampled at rate T_s , giving

$$x_{\delta}(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$
(1)

The CT spectrum for the sampled signal (using FT of the product) is:

$$\mathcal{F}\{x_{\delta}(t)\} = X_{\delta}(j\omega) \tag{2}$$

$$= X(j\omega) * \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{k2\pi}{T_s})$$
(3)

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j\omega - \frac{k2\pi}{T_s})$$
(4)

or, taking FT of each term in the sum for $x_{\delta}(t)$ (eq: 1):

$$X_{\delta}(j\omega) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega nT_s}$$

To calculate the frequency normalized DTFT, let $\Omega = \omega T_s$. (Check that $\omega = \frac{2\pi}{T_s}$ corresponds to $\Omega = 2\pi$.) Thus

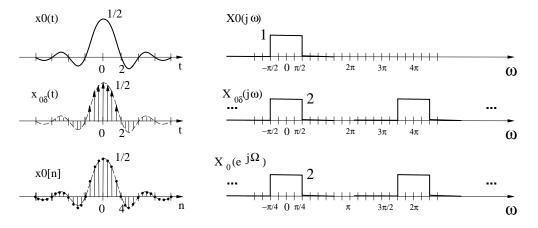
$$X_{\delta}(\frac{j\Omega}{T_s}) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\Omega n}$$

This is just the DTFT analysis equation. The frequency normalization can cause some issues when the FT contains $\delta(\omega)$ as shown in Example 1 and 2 below.

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = X_{\delta}(\frac{j\Omega}{T_s})$$

Time Domain

Frequency Domain



Example 0

Let $x_0(t) = \frac{\sin \pi t/2}{\pi t}$, then $X_0(j\omega) = \Pi(\frac{\omega}{\pi})$. With sampling rate T_0 the sampled signal $x_0(nT_0) = \frac{\sin \pi n T_0/2}{\pi n T_0}$ has FT:

$$X_{0\delta}(j\omega) = X_0(j\omega) * \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{k2\pi}{T_0})$$
(5)

$$= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} X_0 (j\omega - \frac{k2\pi}{T_0}) \tag{6}$$

$$= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \prod(\frac{\omega - \frac{k2\pi}{T_0}}{\pi})$$
(7)

Substituting $\omega = \Omega/T_0$, to normalize frequency range, eq. 7 becomes

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$$X_{0\delta}(j\frac{\Omega}{T_0}) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \Pi(\frac{\frac{\Omega}{T_0} - \frac{k2\pi}{T_0}}{\pi})$$
(8)

$$= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \Pi(\frac{\Omega - k2\pi}{\pi T_0})$$
(9)

We can show that the FT of the sampled signal $x_{0\delta}(t)$ is the same as the DTFT of $x_0[n]$, that is that $X_{0\delta}(j\frac{\Omega}{T_0}) = X_0(e^{j\Omega})$. The DTFT is

$$X_0(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_0[n] e^{-j\Omega n}$$
(10)

$$= \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \frac{\sin \frac{\pi T_0}{2} n}{\pi n} e^{-j\Omega n}$$
(11)

$$= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \prod(\frac{\Omega - 2\pi k}{\pi T_0})$$
(12)

Where $x_0[n]$ was found from the IDTFT of $X_0(e^{j\Omega})$ over one period which is $\frac{1}{T_0}\Pi(\frac{\Omega}{\pi T_0})$. Thus

$$x_0[n] = \frac{1}{2\pi} \int_{-\frac{\pi T_0}{2}}^{\frac{\pi T_0}{2}} \frac{1}{T_0} e^{j\Omega n} d\Omega = \frac{1}{2\pi T_0} \frac{2\sin\frac{\pi T_0}{2}n}{n} = \frac{\sin\frac{\pi T_0}{2}n}{\pi n T_0}$$
(13)

We have shown that $X_{0\delta}(j\frac{\Omega}{T_0}) = X_0(e^{j\Omega})$. Also the result agrees with Table 5.2. This example is shown in the figure above, with $T_0 = 1/2$.

Example 1

Let $x_1(t) = \cos(2\pi t)$ with sampling rate $T_1 = \frac{1}{8}$. Then $X_1(j\omega) = \pi(\delta(\omega - 2\pi) + \delta(\omega + 2\pi))$. The sampled signal has FT:

$$X_{1\delta}(j\omega) = \frac{1}{T_1} \sum_{k=-\infty}^{\infty} X(j\omega - \frac{k2\pi}{T_1})$$
(14)

$$= 8\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi - \frac{k2\pi}{T_1}) + \delta(\omega + 2\pi - \frac{k2\pi}{T_1})$$
(15)

Substituting $\omega = \Omega/T_1$, eq. 15 becomes

$$X_{1\delta}(j\frac{\Omega}{T_1}) = 8\pi \sum_{k=-\infty}^{\infty} \delta(\frac{\Omega}{T_1} - 2\pi - \frac{k2\pi}{T_1}) + \delta(\frac{\Omega}{T_1} + 2\pi - \frac{k2\pi}{T_1})$$
(16)

$$= 8\pi \sum_{k=-\infty}^{\infty} \delta(\frac{\Omega - 2\pi T_1 - k2\pi}{T_1}) + \delta(\frac{\Omega + 2\pi T_1 - k2\pi}{T_1})$$
(17)

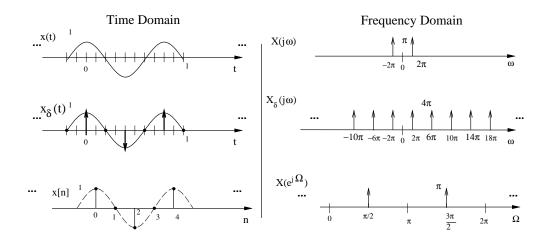
Now since $\delta(at) = \frac{1}{|a|}\delta(t), \, \delta(\frac{\Omega+2\pi T_1-k2\pi}{T_1}) = T_1\delta(\Omega+2\pi T_1-k2\pi).$ Then

$$X_{1\delta}(j\frac{\Omega}{T_1}) = 8\pi T_1 \sum_{k=-\infty}^{\infty} \delta(\Omega + 2\pi T_1 - k2\pi) + \delta(\Omega - 2\pi T_1 - k2\pi)$$
(18)

$$= \pi \sum_{k=-\infty}^{\infty} \delta(\Omega + \frac{\pi}{4} - k2\pi) + \delta(\Omega - \frac{\pi}{4} - k2\pi)$$
(19)

$$= X_{1\delta}(e^{j\Omega}).$$
⁽²⁰⁾

This agrees with Table 5.2 in OW.



Example 2

Let $x_2(t) = \cos(2\pi t)$ with sampling rate $T_2 = \frac{1}{4}$. Then $X_2(j\omega) = \pi(\delta(\omega - 2\pi) + \delta(\omega + 2\pi))$. The sampled signal has FT:

$$X_{2\delta}(j\omega) = 4\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi - \frac{k2\pi}{T_2}) + \delta(\omega + 2\pi - \frac{k2\pi}{T_2})$$
(21)

Substituting $\omega = \Omega/T$, eq. 21 becomes

$$X_{2\delta}(j\frac{\Omega}{T}) = 4\pi T_s \sum_{k=-\infty}^{\infty} \delta(\Omega + 2\pi T_2 - k2\pi) + \delta(\Omega - 2\pi T_2 - k2\pi)$$
(22)

$$= \pi \sum_{k=-\infty}^{\infty} \delta(\Omega + \frac{\pi}{2} - k2\pi) + \delta(\Omega - \frac{\pi}{2} - k2\pi)$$
(23)

$$= X_{2\delta}(e^{j\Omega}). \tag{24}$$

This agrees with Table 5.2 in OW.

Comparing example 1 and example 2, it is seen that the DTFT amplitude does not depend on sample period T_i for this example, due the scaling property of the $\delta()$ function.