EE120 - Fall'15 - Lecture 9 Notes[1]

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Discrete Fourier Transform (DFT)

DFT is applicable to a finite duration sequence $x[n]$ such that $x[n] = 0$ when $n < 0$ and $n \geq N$.

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} & \text{k=0,1,\ldots,N-1} \\ 0 & \text{for other } k. \end{cases} \tag{1}$$

Connection to Discrete Fourier Series

If we define a period-N sequence $\tilde{x}[n]$ by adding $x[n]$ end-to-end:

$$\tilde{x}[n] := x[(n \bmod N)] \tag{2}$$

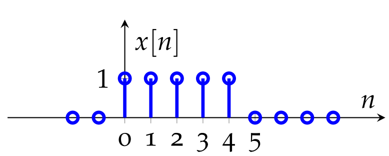
then $X[k] = Na_k$ for $k = 0, 1, \ldots, N-1$:

$$Na_k := X[(k \bmod N)]. \tag{3}$$

Synthesis Equation for DFT:

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} & 0 \leq n \leq N-1 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Example:

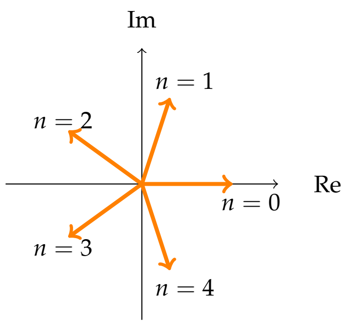


Take $N = 5$ (5-point DFT):

$$X[k] = \sum_{n=0}^{4} e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, 2, 3, 4 \tag{5}$$

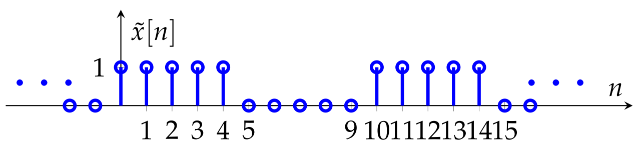
$$= \begin{cases} 5 & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

To see why $X[k] = 0$ for, say $k = 1$, note that (5) is the sum of the following complex numbers:



What if we take $N = 10$ (10-point DFT)?

$X[k] = 10a_k, k = 0, 1, \ldots, 9$ where a_k are FS coefficients of:



From Lecture 3, (FS of rectangular pulse train):

$$X[k] = Na_k = e^{-j\frac{4\pi}{10}k} \frac{\sin\left(\frac{\pi}{2}k\right)}{\sin\left(\frac{\pi}{10}k\right)} \quad k = 0, 1, \ldots, 9. \tag{7}$$

$X[0] = 5, X[2] = X[4] = X[6] = X[8] = 0, X[1] = 1 - j3.0777,$
$X[3] = 1 - j0.7265, X[5] = 1, X[7] = 1 + j0.7265, X[9] = 1 + j3.0777.$

Conjugate Symmetry Property of the DFT:

If $x[n]$ is real, then

$$\boxed{X^*[N-k] = X[k] \quad k = 1, 2, \ldots, N-1.} \tag{8}$$

Example: In the 10-point DFT above, $X[1] = X^*[9]$, $X[3] = X^*[7], \ldots$.

Connection between DFT and DTFT

Recall:

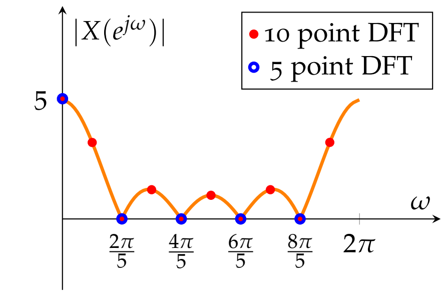
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \tag{9}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, \ldots, N-1 \tag{10}$$

$$\boxed{X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k}} \quad N \text{ samples of DTFT at } \omega = \frac{2\pi}{N}k, \; k = 0, 1, \ldots, N-1. \tag{11}$$

Back to the example:

$$X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)} \tag{12}$$



MATLAB:

`fft([1 1 1 1 1])` for 5-point,

`fft([1 1 1 1 1 0 0 0 0 0])` for 10-point DFT.

The derivation of $X[k] = X(e^{j\omega})\big|_{\omega=\frac{2\pi}{N}k}$ above assumed $N \geq$ duration of $x[n]$. What do we recover from N samples of the DTFT otherwise?

$$x[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\omega}) \tag{13}$$

$$? \overset{DFT}{\longleftrightarrow} X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k} \tag{14}$$

$$\tag{15}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\Big(\underbrace{X(e^{j\omega})\Big|_{\omega=\frac{2\pi}{N}k}}_{=\sum_{m=\infty}^{\infty} x[m]e^{-j\frac{2\pi}{N}km}}\Big)e^{j\frac{2\pi}{N}kn} \quad n = 0,1,\ldots,N-1 \tag{16}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} x[m]e^{j\frac{2\pi}{N}k(n-m)} \tag{17}$$

$$= \sum_{m=\infty}^{\infty} x[m]\Big(\underbrace{\frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-m)}}_{=\begin{cases}1 & \text{if } m = n \bmod N\\ 0 & \text{otherwise}\end{cases}}\Big) \tag{18}$$

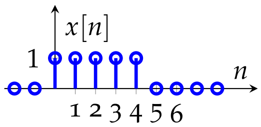
$$= \ldots + x[n-N] + x[n] + x[n+N] + \ldots \tag{19}$$

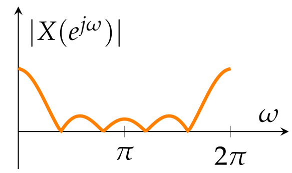
$$\tag{20}$$

$$\boxed{= \sum_{r=-\infty}^{\infty} x[n-rN] \quad 0 \leq n \leq N-1.} \tag{21}$$

$= x[n]$ if $N \geq$ duration of $x[n]$; otherwise, aliasing in time!

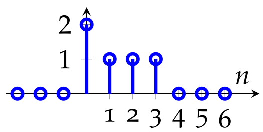
Example:





$N \geq 5$ samples of $X(e^{j\omega})$ correspond to N-point DFT of $x[n]$.
$N = 4$ samples of $X(e^{j\omega})$ correspond to 4-point DFT of:

$$\sum_{r=-\infty}^{\infty} x[n-4r] \quad 0 \leq n \leq 3. \tag{22}$$



DFT Makes Convolution Easy

$$y[n] = \underbrace{h[n]}_{\substack{\text{FIR with} \\ \text{duration} = P}} * \underbrace{x[n]}_{\substack{\text{input sequence} \\ \text{with duration} = L}} \tag{23}$$

Set $N \geq L + P - 1$ (duration of $y[n]$). Pad zeros in $h[n]$ and $x[n]$ to make them duration $= N$. Take their N-point DFT[2]to find $H[k]$ and $X[k]$. Take inverse DFT[3]of $H[k] \cdot X[k]$ to obtain $y[n]$.

[2] e.g., using `fft` in MATLAB

[3] `ifft` in MATLAB

2D DFT

Analysis Equation: For $0 \leq k_1 \leq N_1 - 1$ and $0 \leq k_2 \leq N_2 - 1$:

$$X[k_1, k_2] = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x[n_1, n_2] e^{-j\frac{2\pi}{N_1}k_1 n_1} e^{-j\frac{2\pi}{N_2}k_2 n_2} \tag{24}$$

Synthesis Equation: For $0 \leq n_1 \leq N_1 - 1$ and $0 \leq n_2 \leq N_2 - 1$:

$$x[n_1, n_2] = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X[k_1, k_2] e^{j\frac{2\pi}{N_1}k_1 n_1} e^{\frac{2\pi}{N_2}k_2 n_2} \tag{25}$$

Reading for Interested Students

Chapter 27: Data Compression in *The Scientist and Engineer's Guide to Digital Signal Processing* (`www.dspguide.com`). See in particular Figures 27-9, 27-10, 27-11, 27-12, 27-15.