EE120 - Fall'15 - Lecture 8 Notes[1]

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Phase Distortion in LTI Systems

Section 6.2 *in* Oppenheim & Willsky

The phase of the Fourier Transform $\measuredangle X(e^{j\omega})$ is as important as the magnitude $|X(e^{j\omega})|$ in describing the features of the signal $x[n]$.

Example: Recall $x[-n] \longleftrightarrow X(e^{-j\omega}) = X^*(e^{j\omega})$ when $x[n]$ is real. Since $|X(e^{j\omega})| = |X^*(e^{j\omega})|$, the DTFT of $x[n]$ and $x[-n]$ differ only by their phase. This shows that phase difference alone can distinguish two signals significantly.

Consider an LTI system whose frequency response can be written as:

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega}$$

where $A(e^{j\omega})$ is *real* and *nonnegative*. Such a system is called "linear phase" because the phase $\measuredangle H(e^{j\omega}) = -\alpha\omega$ is a linear function of ω.

Linear phase filters are desirable because each frequency component of a signal passing through them is delayed by the same duration, α:

$$e^{j\omega n} \rightarrow \boxed{H(e^{j\omega})} \rightarrow H(e^{j\omega})e^{j\omega n} = A(e^{j\omega})e^{j\omega(n-\alpha)}$$

By contrast, an LTI system whose phase $\measuredangle H(e^{j\omega})$ depends nonlinearly on ω delays each frequency component differently and can cause severe distortion. See page 3 for an example of such phase distortion.

The linear phase property is very restrictive in practice. The relaxed version below maintains the essential benefits and is easy to achieve in FIR filter design:

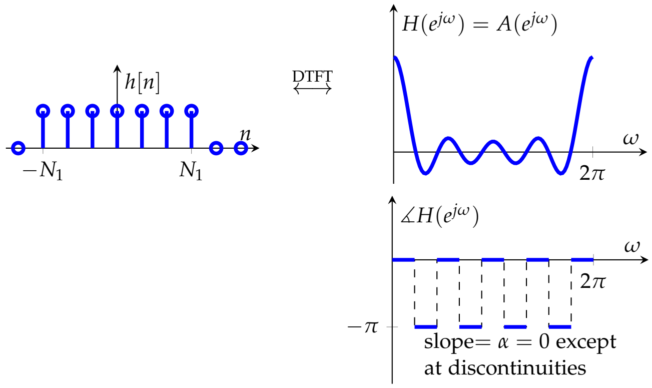
Generalized Linear Phase Systems

An LTI system with frequency response $H(e^{j\omega})$ is called "generalized linear phase" if we can find a real-valued function $A(e^{j\omega})$ and constants α, β, such that:

$$\boxed{H(e^{j\omega}) = \underbrace{A(e^{j\omega})}_{\text{real, but sign change allowed}} e^{-j\alpha\omega + j\beta}} \tag{1}$$

Note that $\measuredangle H(e^{j\omega}) = \beta - \alpha\omega$ for each ω such that $A(e^{j\omega}) > 0$.
If the sign of $A(e^{j\omega})$ changes at a frequency ω, then $\measuredangle H(e^{j\omega})$ changes discontinuously by π.

Example: If $h[-n] = h[n]$ (even symmetric), then $H(e^{j\omega})$ is real. We can take $A(e^{j\omega}) = H(e^{j\omega})$, $\alpha = \beta = 0$.



Example: If $h[n] = h_0[n - N_1]$ where $h_0[n]$ is even symmetric, then

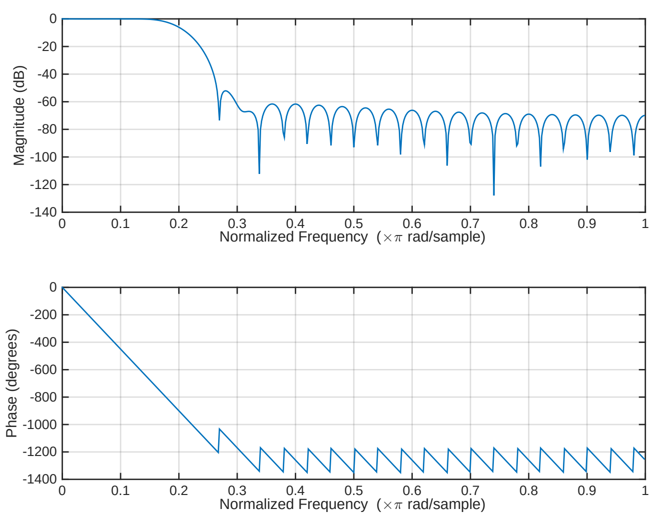
$$H(e^{j\omega}) = H_0(e^{j\omega})e^{-j\omega N_1}. \tag{2}$$

Since $H_0(e^{j\omega})$ is real, take $A(e^{j\omega}) = H_0(e^{j\omega})$, $\alpha = N_1$, $\beta = 0$.

The windowed FIR filters in the last lecture have this form, therefore they are generalized linear phase:

$$h_0[n] = \underbrace{\frac{\sin \omega_c n}{\pi n}}_{\substack{\text{impulse resp.} \\ \text{of ideal LPF}}} \cdot \underbrace{w[n]}_{\substack{\text{window} \\ \text{(rectangular,} \\ \text{Hamming, } etc.\text{)}}} \quad \text{(even symmetric).} \tag{3}$$

Frequency response of a Hamming windowed filter from last lecture:



Example: If $h[-n] = -h[n]$ (odd symmetric) then $H(e^{j\omega})$ is purely imaginary. Let $A(e^{j\omega}) = -jH(e^{j\omega}) = H(e^{j\omega})e^{-j\pi/2}$ (real).

$$H(e^{j\omega}) = A(e^{j\omega})e^{j\pi/2} \qquad \rightarrow \qquad \alpha = 0, \beta = \pi/2. \tag{4}$$

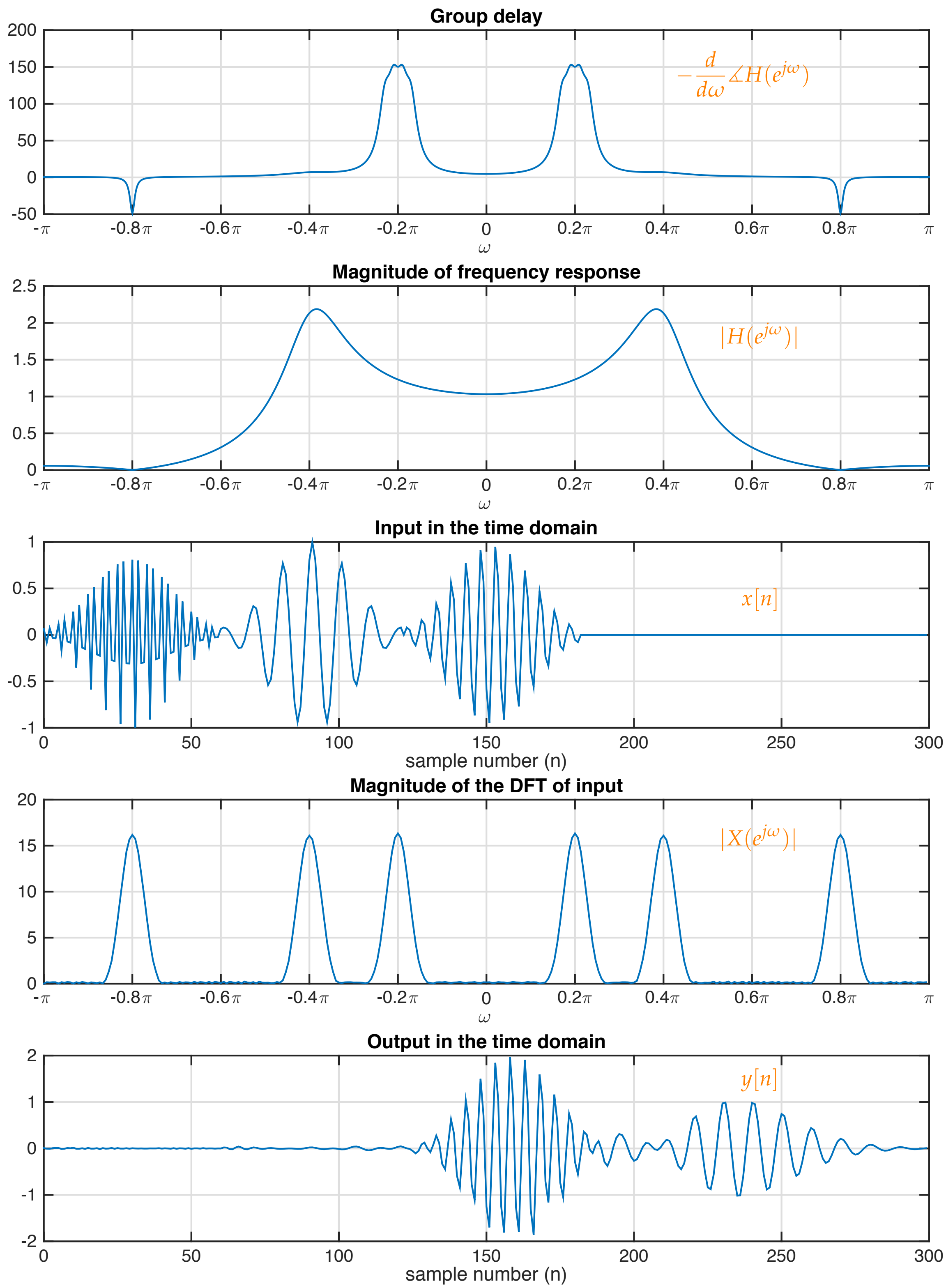


Figure 1: Phase distortion illustrated on a signal $x[n]$ (middle plot) with three dominant frequency components (shown in the plot of $|X(e^{j\omega})|$ underneath). This signal is applied to a LTI system with highly nonlinear phase, as seen from the plot for -$\frac{d}{d\omega}\measuredangle H(e^{j\omega})$ (top). In the output $y[n]$ (bottom), the order of the the low and middle frequency components are swapped because the low frequency component incurred a large delay. The high frequency component ($\omega = 0.8\pi$) is filtered out because $H(e^{j0.8\pi}) = 0$ (the plot second from the top). See Section 5.1.2 in Oppenheim & Schafer, Discrete-Time Signal Processing, 3rd ed., Prentice Hall, for the construction of the LTI system in this example.

Two Dimensional (2D) Fourier Transform

2D CTFT Analysis Equation:

$$\boxed{X(j\omega_1, j\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1,t_2)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1 dt_2} \tag{5}$$

2D CTFT Synthesis Equation:

$$\boxed{x(t_1,t_2) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(j\omega_1, j\omega_2)e^{j\omega_1 t_1}e^{j\omega_2 t_2}d\omega_1 d\omega_2} \tag{6}$$

2D DTFT Analysis Equation:

$$\boxed{X(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x[n_1,n_2]e^{-j\omega_1 n_1}e^{-j\omega_2 n_2}} \tag{7}$$

Note that this is periodic with period $(2\pi, 2\pi)$:

$$X(e^{j\omega_1}, e^{j\omega_2}) = X(e^{j(\omega_1+2\pi)}, e^{j\omega_2}) = X(e^{j\omega_1}, e^{j(\omega_2+2\pi)}).$$

2D DTFT Synthesis Equation:

$$\boxed{x[n_1,n_2] = \frac{1}{(2\pi)^2}\int_{2\pi}\int_{2\pi} X(e^{j\omega_1}, e^{j\omega_2})e^{j\omega_1 n_1}e^{j\omega_2 n_2}d\omega_1 d\omega_2} \tag{8}$$

Absolute integrability/summability conditions for convergence:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x(t_1,t_2)|dt_1 dt_2 < \infty \quad \text{(continuous time)} \tag{9}$$

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} |x[n_1,n_2]| < \infty \quad \text{(discrete time).} \tag{10}$$

<u>Example:</u> $x[n_1,n_2] = \delta[n_1,n_2] := \delta[n_1]\delta[n_2]$.

$$X(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} \delta[n_1,n_2]e^{-j\omega_1 n_1}e^{-j\omega_2 n_2} = e^{-j\omega_1 0}e^{-j\omega_2 0} = 1$$

<u>Example:</u> $x[n_1,n_2] = a^{n_1}b^{n_2}u[n_1,n_2], \quad |a| < 1, \ |b| < 1.$

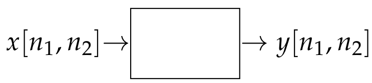
$$\begin{aligned} X(e^{j\omega_1}, e^{j\omega_2}) &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a^{n_1}b^{n_2}e^{-j\omega_1 n_1}e^{-j\omega_2 n_2} \\ &= \sum_{n_1=0}^{\infty} a^{n_1}e^{-j\omega_1 n_1}\sum_{n_2=0}^{\infty} b^{n_2}e^{-j\omega_2 n_2} \\ &= \frac{1}{1-ae^{-j\omega_1}}\frac{1}{1-be^{-j\omega_2}} \end{aligned}$$

Separability Property of the 2D DTFT:

If $x[n_1, n_2] = x_1[n_1]x_2[n_2]$ then $X(e^{j\omega_1}, e^{j\omega_2}) = X_1(e^{j\omega_1})X_2(e^{j\omega_2})$ as in the examples above. A similar property holds for the 2D CTFT.
Proof:

$$\begin{aligned} X(e^{j\omega_1}, e^{j\omega_2}) &= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x_1[n_1]x_2[n_2]e^{-j\omega_1 n_1}e^{-j\omega_2 n_2} \\ &= \underbrace{\sum_{n_1=-\infty}^{\infty} x_1[n_1]e^{-j\omega_1 n_1}}_{=X_1(e^{j\omega_1})} \underbrace{\sum_{n_2=-\infty}^{\infty} x_2[n_2]e^{-j\omega_2 n_2}}_{=X_2(e^{j\omega_2})} \end{aligned}$$

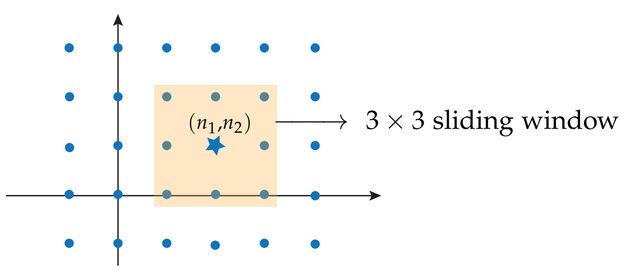
2D Systems



When the input is $\delta[n_1, n_2]$ the output is called the *impulse response* and denoted $h[n_1, n_2]$ as in 1D systems.

Example: 2D moving average filter

$$y[n_1, n_2] = \frac{1}{9} \sum_{k_1=-1}^{1} \sum_{k_2=-1}^{1} x[n_1 - k_1, n_2 - k_2]$$



$$h[n_1, n_2] = \begin{cases} \frac{1}{9} & -1 \le n_1 \le 1 \text{ and } -1 \le n_2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

2D Convolution:

If the system is linear shift-invariant, then:

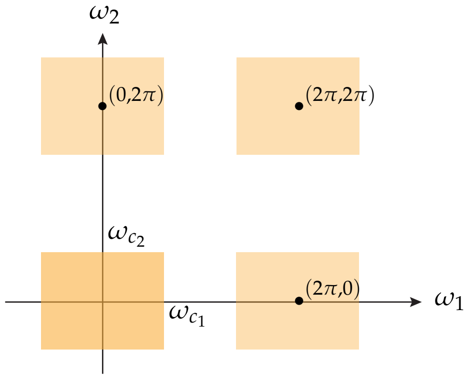
$$\begin{aligned} y[n_1, n_2] &= h[n_1, n_2] * x[n_1, n_2] \\ &= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} h[m_1, m_2]x[n_1 - m_1, n_2 - m_2] \\ &= \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} x[m_1, m_2]h[n_1 - m_1, n_2 - m_2]. \end{aligned}$$

Convolution Property of the 2D DTFT

$$\boxed{h[n_1,n_2] * x[n_1,n_2] \;\longleftrightarrow\; H(e^{j\omega_1},e^{j\omega_2})X(e^{j\omega_1},e^{j\omega_2})} \qquad (11)$$

Example: 2D separable ideal low pass filter

$H(e^{j\omega_1},e^{j\omega_2}) = 1$ in the shaded regions of the (ω_1,ω_2)-plane below and $= 0$ otherwise:



We can write this frequency response as:

$$H(e^{j\omega_1},e^{j\omega_2}) = H_1(e^{j\omega_1})H_2(e^{j\omega_2})$$

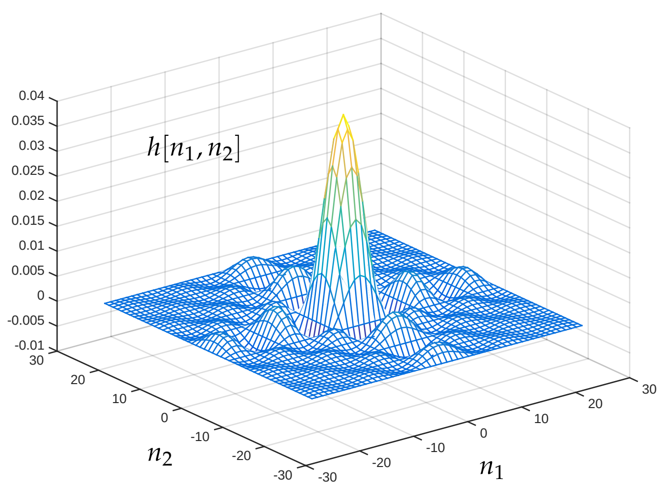
where

$$H_i(e^{j\omega_i}) = \begin{cases} 1 & |\omega_i| \le \omega_{c_i} \\ 0 & \omega_{c_i} < |\omega_i| \le \pi \end{cases} \qquad i = 1,2.$$

Then, from the separability property,

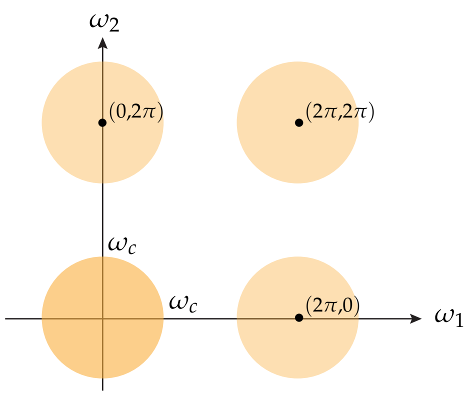
$$h[n_1,n_2] = \frac{\sin\omega_{c_1}n_1}{\pi n_1}\frac{\sin\omega_{c_2}n_2}{\pi n_2}$$

which is depicted below for $\omega_{c_1} = \omega_{c_2} = 0.2\pi$.



Example: 2D circularly symmetric ideal low pass filter

$H(e^{j\omega_1}, e^{j\omega_2}) = 1$ in the shaded regions of the (ω_1, ω_2)-plane below and $= 0$ otherwise:



In the region $[-\pi, \pi] \times [-\pi, \pi]$, this can be expressed as:

$$H(e^{j\omega_1}, e^{j\omega_2}) = \begin{cases} 1 & \sqrt{\omega_1^2 + \omega_2^2} \leq \omega_c \\ 0 & \sqrt{\omega_1^2 + \omega_2^2} > \omega_c. \end{cases}$$

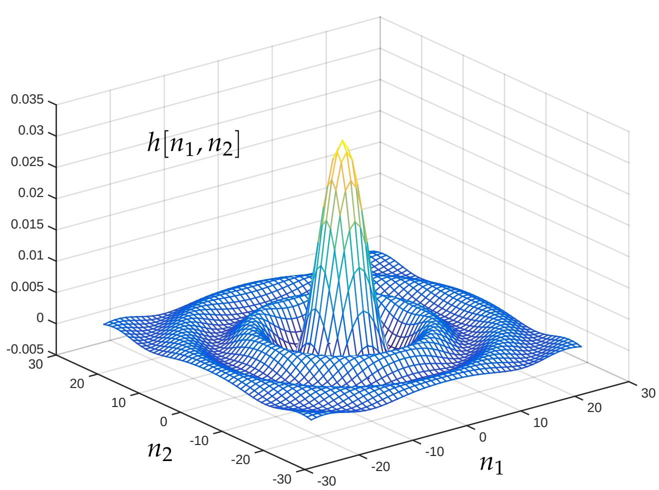
The 2D DTFT Synthesis Equation yields:

$$h[n_1, n_2] = \frac{\omega_c}{2\pi\sqrt{n_1^2 + n_2^2}} J_1\left(\omega_c\sqrt{n_1^2 + n_2^2}\right)$$

where $J_1(\cdot)$ is the Bessel function of the first kind and first order.[2]

Note that $h[n_1, n_2]$ is not separable. However, like the frequency response $H(e^{j\omega_1}, e^{j\omega_2})$, it exhibits circular symmetry. See the figure below for a depiction of $h[n_1, n_2]$ for $\omega_c = 0.2\pi$.

[2] See mathworld.wolfram.com for a description of Bessel functions of the first kind. The Matlab command to evaluate $J_1(\cdot)$ is `besselj(1,·)` where the first argument specifies the order.



Projection-Slice Theorem

Consider the following "projection" of the 2D function $x(t_1, t_2)$ along the t_2 axis:

$$x_0(t_1) \triangleq \int_{-\infty}^{\infty} x(t_1, t_2) dt_2.$$

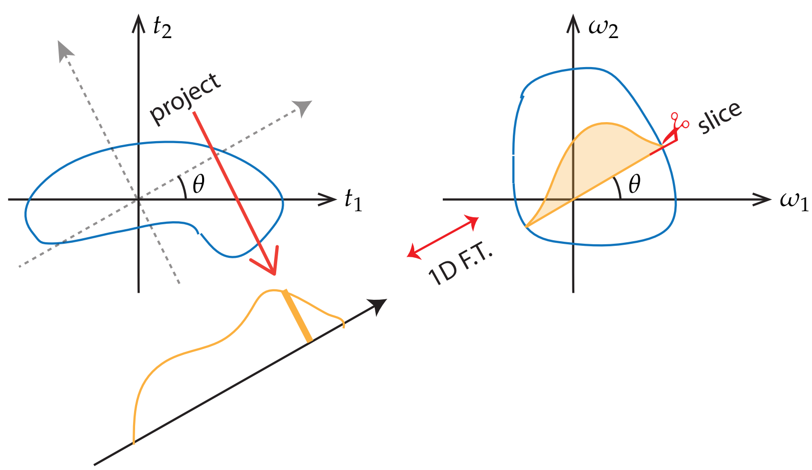
Then, the 1D CTFT of $x_0(t_1)$ is related to the 2D CTFT of $x(t_1, t_2)$ by:

$$X_0(j\omega_1) = X(j\omega_1, j\omega_2)|_{\omega_2=0}.$$

This is because:

$$\begin{aligned} X(j\omega_1, j\omega_2)|_{\omega_2=0} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1, t_2) e^{-j\omega_1 t_1} e^{-j0t_2} dt_1 dt_2 \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t_1, t_2) dt_2 \right) e^{-j\omega_1 t_1} dt_1 \\ &= \int_{-\infty}^{\infty} x_0(t_1) e^{-j\omega_1 t_1} dt_1 = X_0(j\omega_1). \end{aligned}$$

A generalization of this property to projections along any direction is known as the *projection-slice theorem* and is illustrated in the figure below. (Projection along the t_2 axis above is the special case $\theta = 0$.)



This theorem is crucial in tomography where one collects projections at many angles about a 2D object. The 1D Fourier Transform of each such "shadow" corresponds to a slice of the 2D Fourier Transform. One can thus obtain the 2D Fourier Transform by combining these slices and then reconstruct $x(t_1, t_2)$ from the synthesis equation.