

# EE120 - Fall'15 - Lecture 7 Notes<sup>1</sup>

Murat Arcak

21 September 2015

<sup>1</sup> Licensed under a [Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License](https://creativecommons.org/licenses/by-nc-sa/4.0/).

## Discrete Time Fourier Transform (DTFT) Continued

### Finding the Frequency Response from a Difference Equation

Section 5.8 in Oppenheim & Willsky

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k] \quad (1)$$

Substitute  $x[n] = \delta[n]$ , and  $y[n] = h[n]$ :

$$\sum_{k=0}^N a_k h[n-k] = \sum_{k=0}^M b_k \delta[n-k] \quad (2)$$

Take the Fourier Transform of both sides (recall that  $\delta[n] \leftrightarrow 1$ ):

$$\left( \sum_{k=0}^N a_k e^{-j\omega k} \right) H(e^{j\omega}) = \sum_{k=0}^M b_k e^{-j\omega k} \quad (3)$$

$$H(e^{j\omega}) = \frac{\sum_{k=0}^M b_k e^{-j\omega k}}{\sum_{k=0}^N a_k e^{-j\omega k}} \quad (4)$$

We can find the impulse response  $h[n]$  from the inverse Fourier transform of  $H(e^{j\omega})$ :

Example:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n] \quad (5)$$

Frequency response:

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}} = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})}$$

Partial fraction expansion:

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}$$

Thus, the impulse response is:

$$h[n] = 4 \left( \frac{1}{2} \right)^n u[n] - 2 \left( \frac{1}{4} \right)^n u[n]$$

Example: Describe the LTI system with impulse response  $h[n] = \alpha^n u[n]$ ,  $|\alpha| < 1$ , with a difference equation.

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \quad (6)$$

which is (4) with  $b_0 = 1$ ,  $\alpha_0 = 1$ , and  $a_1 = -\alpha$ . Thus,

$$y[n] - \alpha y[n-1] = x[n].$$

Example: Find the difference equation describing a LTI system whose impulse response is:

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{2} \left(\frac{1}{4}\right)^n u[n].$$

*Convolution Property of DTFT*

Section 5.4 in Oppenheim & Willsky

$$y[n] = h[n] * x[n] \longleftrightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \quad (7)$$

Example:

$$h[n] = \alpha^n u[n] \quad |\alpha| < 1 \quad \leftrightarrow \quad H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

$$x[n] = \beta^n u[n] \quad |\beta| < 1 \quad \leftrightarrow \quad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}$$

If  $\alpha \neq \beta$ , employ partial fraction expansion:

$$Y(e^{j\omega}) = \frac{A}{1 - \alpha e^{-j\omega}} + \frac{B}{1 - \beta e^{-j\omega}},$$

with  $A + B = 1$  and  $A\beta + B\alpha = 0$ . Then,  $A = \frac{\alpha}{\alpha - \beta}$  and  $B = \frac{-\beta}{\alpha - \beta}$ .  
Then,

$$y[n] = \frac{\alpha}{\alpha - \beta} \alpha^n u[n] - \frac{\beta}{\alpha - \beta} \beta^n u[n] = \frac{1}{\alpha - \beta} (\alpha^{n+1} - \beta^{n+1}) u[n]$$

If  $\alpha = \beta$ , then:

$$Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^2} = \frac{\overbrace{e^{j\omega}}^{\text{time-shift}}}{-\alpha j} \underbrace{\frac{d}{d\omega} \left( \frac{1}{1 - \alpha e^{-j\omega}} \right)}_{\leftrightarrow \frac{1}{j} n \alpha^n u[n]^2}$$

<sup>2</sup> by the differentiation property:

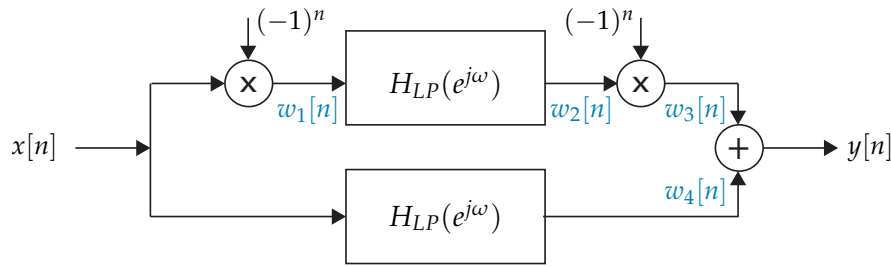
$$nx[n] \leftrightarrow j \frac{dX(e^{j\omega})}{d\omega}$$

Then,

$$y[n] = \frac{1}{\alpha} (n+1) \alpha^{n+1} u[n+1] = (n+1) \alpha^n u[n+1] = (n+1) \alpha^n u[n]$$

where we replaced  $u[n+1]$  with  $u[n]$  since  $(n+1)\alpha^n = 0$  for  $n = -1$ .

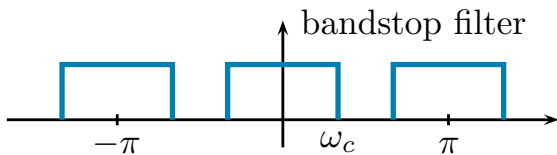
Example: Determine the function performed by the block diagram below where  $H_{LP}$  is a low-pass filter with cutoff frequency  $\omega_c < \pi/2$ .



$$\begin{aligned} W_1(e^{j\omega}) &= X(e^{j(\omega-\pi)}) \\ W_2(e^{j\omega}) &= H_{LP}(e^{j\omega})X(e^{j(\omega-\pi)}) \\ W_3(e^{j\omega}) &= H_{LP}(e^{j(\omega-\pi)})X(e^{j\omega}) \\ W_4(e^{j\omega}) &= H_{LP}(e^{j\omega})X(e^{j\omega}) \end{aligned}$$

Adding  $W_3(e^{j\omega})$  and  $W_4(e^{j\omega})$ :

$$Y(e^{j\omega}) = \underbrace{(H_{LP}(e^{j\omega}) + H_{LP}(e^{j(\omega-\pi)}))}_{\text{bandstop filter}} X(e^{j\omega}) \tag{8}$$



Multiplication Property

$$x_1[n]x_2[n] \longleftrightarrow \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \tag{9}$$

Section 5.5 in Oppenheim & Willsky

Proof: Apply synthesis equation to the right-hand side:

<sup>3</sup> "periodic convolution"

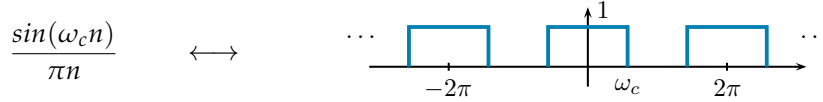
$$\begin{aligned} & \frac{1}{2\pi} \int_{2\pi} \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) \underbrace{\frac{1}{2\pi} \int_{2\pi} X_2(e^{j(\omega-\theta)})e^{j\omega n} d\omega d\theta}_{=e^{j\theta n}x_2[n] \text{ } ^4} \\ &= x_2[n] \underbrace{\frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})e^{j\theta n} d\theta}_{=x_1[n]} = x_1[n]x_2[n]. \end{aligned}$$

<sup>4</sup> from the frequency shift property

Example: 
$$x[n] = \underbrace{\frac{\sin(\frac{3\pi}{4}n)}{\pi n}}_{\triangleq x_1[n]} \cdot \underbrace{\frac{\sin(\frac{\pi}{2}n)}{\pi n}}_{\triangleq x_2[n]}$$

Interpret the value at  $n = 0$  as  $x[0] = x_1[0]x_2[0] = \frac{3}{4} \cdot \frac{1}{2}$

Use the multiplication property to calculate  $X(e^{j\omega})$ . First note that:



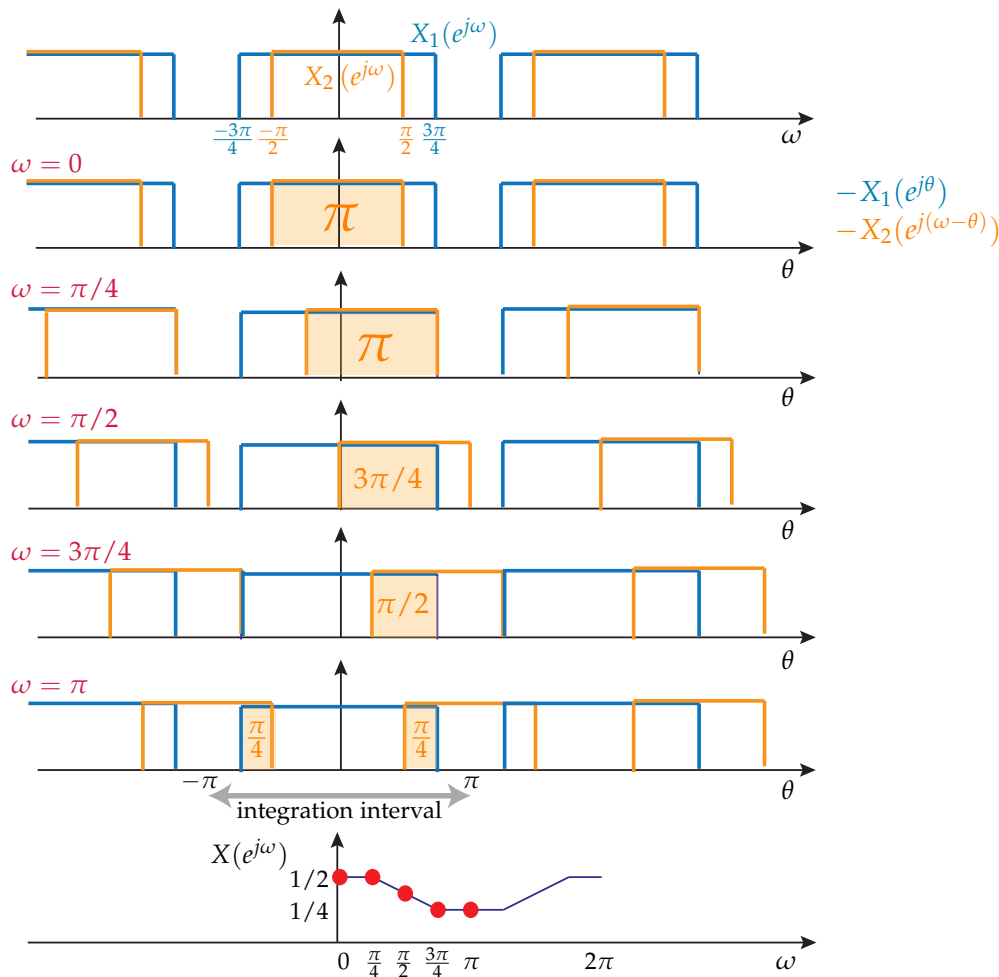
Easy to show by applying the synthesis equation:

$$\frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi} \frac{1}{jn} e^{j\omega n} \Big|_{-\omega_c}^{\omega_c} = \frac{1}{\pi n} \frac{1}{2j} \underbrace{(e^{j\omega_c n} - e^{-j\omega_c n})}_{= \sin \omega_c n}$$

By the multiplication property,

$$X(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta \tag{10}$$

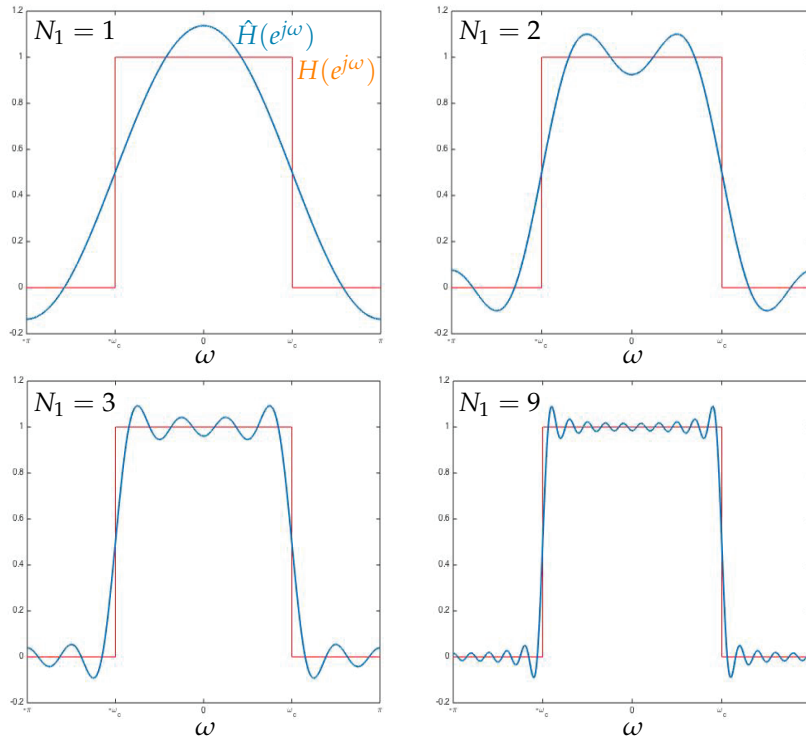
i.e. periodic convolution of rectangular pulses in frequency domain:





### Gibbs Phenomenon and Tapered Windows

Note from the figure below that  $\hat{H}(e^{j\omega})$  exhibits oscillations near the discontinuities of  $H(e^{j\omega})$  and their amplitudes do not decrease as  $N_1$  is increased. This is known as the Gibbs Phenomenon.



These oscillations are caused by the sizable side lobes of  $W(e^{j\omega})$  (high frequency components) which are due to the abrupt change from 0 to 1 in  $w[n]$ .

“Tapered” windows mitigate this problem, e.g., the triangular (Bartlett) window:

$$w[n] = \begin{cases} 1 - \frac{|n|}{N_1} & \text{if } |n| \leq N_1 \\ 0 & \text{otherwise.} \end{cases}$$

Other tapered windows exist (Hanning, Hamming, Blackman, etc.) and are depicted in Figure 1. Although the differences between these windows may not be appreciable in time domain, their Fourier Transforms have significant differences as shown in Figure 2. Note the tradeoff between main lobe width & side lobe amplitude in Figure 2.

must be small for  
sharp transition  
from passband to  
stopband

must be small to  
reduce ripples

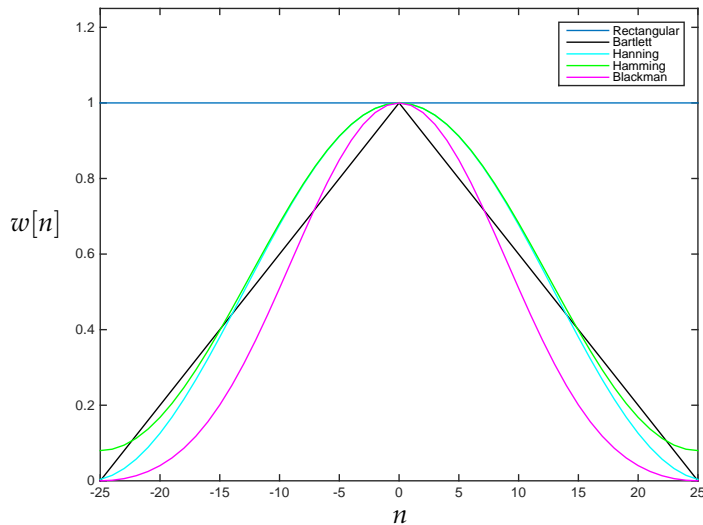


Figure 1: Tapered windows of type Bartlett, Hanning, Hamming, and Blackman for  $N_1 = 25$  superimposed.

### Summary

To obtain a FIR filter truncate the ideal filter's impulse response  $h[n]$  with one of the window functions  $w[n]$ :

$$\hat{h}[n] = h[n]w[n].$$

The new impulse response is zero outside of  $n \in \{-N_1, \dots, N_1\}$  but not yet causal. To make it causal, shift to the right by  $N_1$ :

$$\hat{h}[n - N_1] \longleftrightarrow e^{-j\omega N_1} \hat{H}(e^{j\omega})$$

which does not change the magnitude of the frequency response, only the phase. Finally,  $\hat{h}[n]$  must be scaled by a constant to obtain  $\sum_n \hat{h}[n] = 1$ , so the dc gain is  $\hat{H}(e^{j0}) = 1$ .

FIR implementation:

$$y[n] = b_0x[n] + b_1x[n - 1] + \dots + b_Mx[n - M]$$

where  $b_0, \dots, b_M$  are the impulse response coefficients:  $b_n = \hat{h}[n - N_1]$ .

MATLAB command for design:

`fir1(M,  $\omega_c$ , w)` returns a vector of the coefficients  $b_0, \dots, b_M$  above  $M$ ; filter order ( $2N_1$  above)

$\omega_c$ : desired cutoff frequency divided by  $\pi$  (so the frequency range  $[0, \pi]$  is normalized to  $[0, 1]$ )

**w**: vector of length  $M + 1$  for the values of the window function  $w[n]$ ; enter `boxcar(M + 1)` for a rectangular window, and `bartlett(M + 1)`, `hamming(M + 1)`, `hanning(M + 1)`, `blackman(M + 1)` for others.

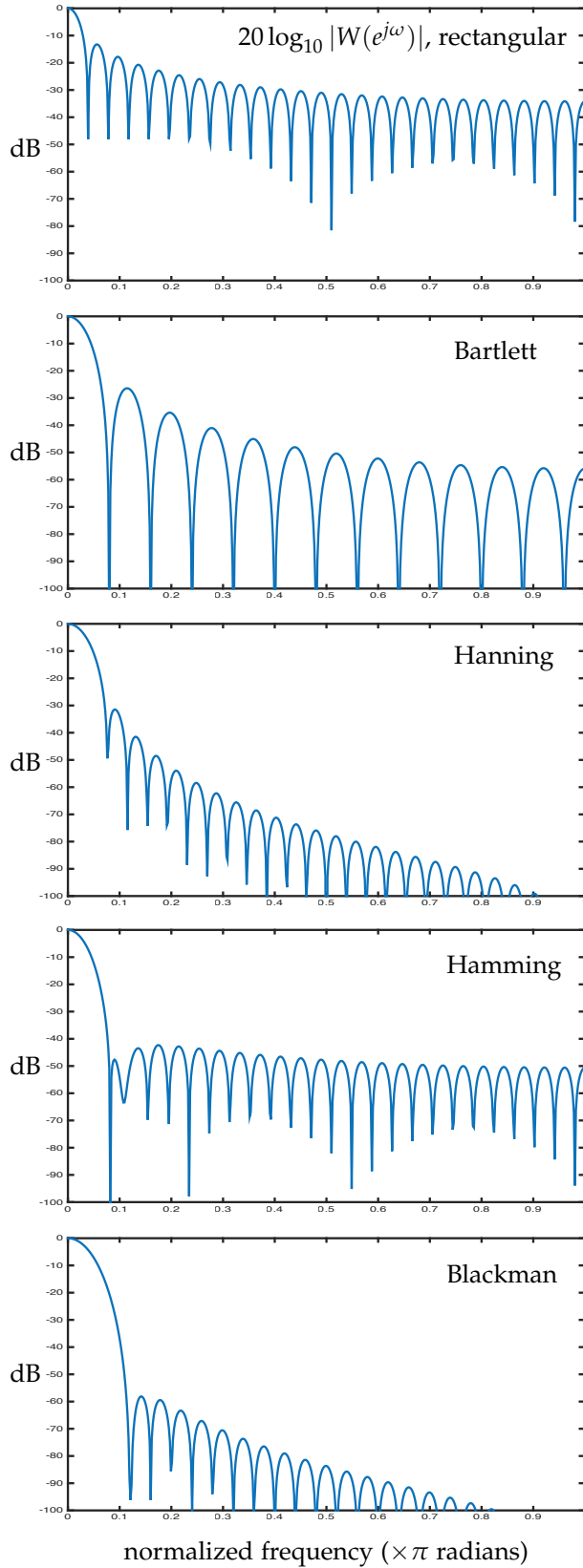
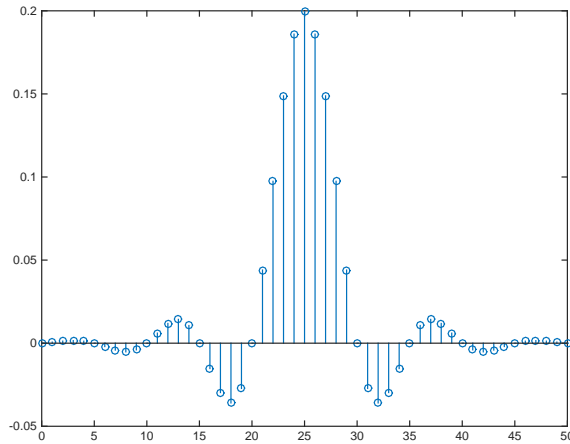


Figure 2: The magnitude of the Fourier Transform  $W(e^{j\omega})$  (in dB) for the rectangular, Bartlett, Hanning, Hamming, and Blackman windows. Here each window function in Figure 1 is scaled such that  $\sum_n w[n] = 1$ , so  $W(e^{j0}) = 1 = 0$  dB. Note that the tapered windows progressively reduce the side lobe amplitudes in the order they are presented. This has the desired effect of reducing ripples in the frequency response of the truncated filter. However, the main lobe width increases which has a negative effect: the transition from passband to stopband will be slower for the filter truncated with the respective window.



Example: `B=fir1(50,0.2,hamming(51))` returns the coefficients of a FIR low pass filter of order  $M = 50$  with cutoff frequency  $0.2\pi$ , truncated with a Hamming window.

`stem(0:50,B)` plots the impulse response:



`freqz(B)` plots the frequency response:

