EE120 - Fall'15 - Lecture 6 Notes[1]

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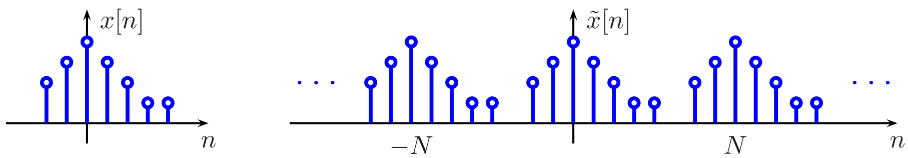
Discrete Time Fourier Transform (DTFT)

Chapter 5 *in* Oppenheim & Willsky

Given aperiodic signal $x[n]$ of finite duration, construct periodic sequence $\tilde{x}[n]$ for which $x[n]$ is one period:



$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jk\frac{2\pi}{N}n} \tag{1}$$

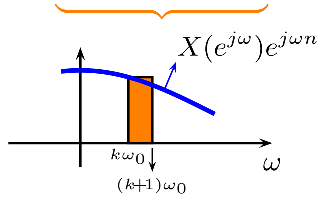
$$Na_k = \sum_{n=-\infty}^{\infty} x[n]e^{-jk\frac{2\pi}{N}n} = \underbrace{\left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right)}_{\substack{\triangleq X(e^{j\omega}) \\ \text{(DTFT)}}}\Bigg|_{\underbrace{\omega = k\frac{2\pi}{N}}_{\substack{\text{more closely spaced} \\ \text{samples as } N\to\infty}}} \tag{2}$$

$$a_k = \frac{1}{N}X(e^{jk\omega_0}), \quad \omega_0 = \frac{2\pi}{N} \tag{3}$$

Furthermore,

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} = \sum_{k=\langle N\rangle}\frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n} \tag{4}$$

$$= \frac{1}{2\pi}\sum_{k=\langle N\rangle} \underbrace{\underbrace{\frac{2\pi}{N}}_{=\omega_0} X(e^{jk\omega_0})e^{jk\omega_0 n}} \tag{5}$$



$$\text{As } N \to \infty: \quad \to \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \tag{6}$$

$$\boxed{\begin{array}{rcll} x[n] & = & \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega & \text{(Synthesis Equation)} \\ X(e^{j\omega}) & = & \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} & \text{(Analysis Equation)} \end{array}} \tag{7}$$

Note that in CT:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega, \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{8}$$

Main difference: Synthesis equation in DT over a period of 2π since $X(e^{j\omega})$ is periodic.

Convergence guaranteed if

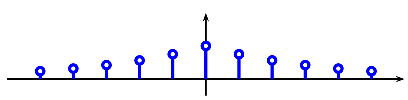
$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \text{(similar to Dirichlet 1 in CT).} \tag{9}$$

Examples:

1) $x[n] = a^n u[n]$, $|a| < 1$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}} \tag{10}$$

2) $x[n] = a^{|n|}$, $|a| < 1$



$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \underbrace{\sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}}_{\sum_{n=1}^{\infty} a^n e^{j\omega n} = \sum_{n=1}^{\infty} a^n e^{-j(-\omega)n} = \sum_{n=0}^{\infty} a^n e^{-j(-\omega)n} - 1} \tag{11}$$

$$= \frac{1}{1-ae^{-j\omega}} + \frac{1}{1-ae^{j\omega}} - 1 \tag{12}$$

$$= \frac{2 - a(e^{j\omega}+e^{-j\omega})}{1 - a(e^{j\omega}+e^{-j\omega}) + a^2} - 1 \tag{13}$$

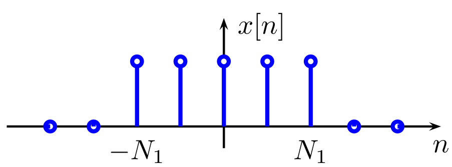
$$= \frac{2-2acos\omega}{1-2acos\omega+a^2} - 1 = \frac{1-a^2}{1+a^2-2acos\omega} \tag{14}$$

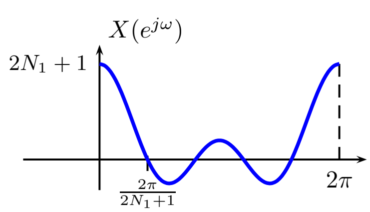
3)

$$x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases} \tag{15}$$

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \begin{cases} \frac{sin(\omega(N_1+1/2))}{sin(\omega/2)} & \omega \neq 0, \\ 2N_1+1 & \omega = 0. \end{cases} \tag{16}$$

Derive this as an exercise.





4)
$$X(e^{j\omega}) = \delta(\omega) \quad -\pi \le \omega \le \pi, \tag{17}$$

or, equivalently:

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) \tag{18}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi} \tag{19}$$

$$x[n] = 1 \leftrightarrow X(e^{j\omega}) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) \tag{20}$$

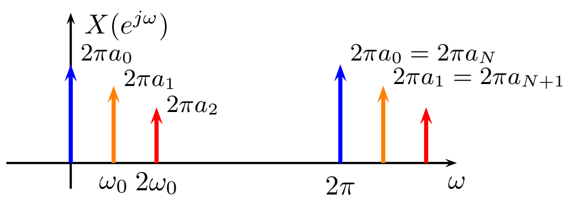
$$x[n] = e^{j\omega_0 n} \leftrightarrow X(e^{j\omega}) = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l) \text{ [2]} \tag{21}$$

[2] from the frequency shift property below

Fourier Transform of Periodic Signals

Section 5.2 *in* Oppenheim & Willsky

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n} \leftrightarrow X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi}{N}k) \tag{22}$$



Properties of DTFT

Section 5.3 *in* Oppenheim & Willsky

Time Shift:

$$\boxed{x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})} \tag{23}$$

Frequency Shift:

$$\boxed{e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})} \tag{24}$$

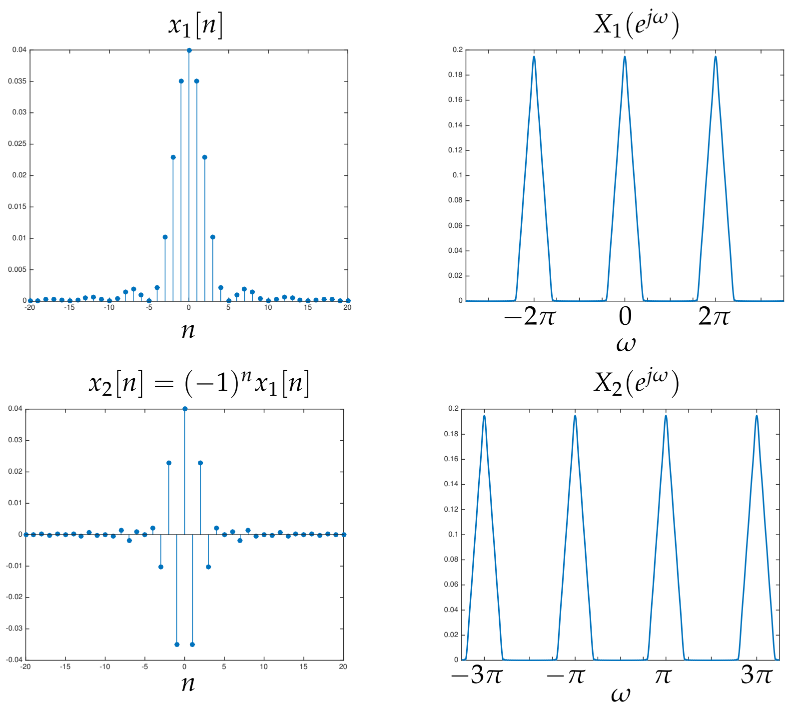
As a special case let $\omega_0 = \pi$ and note that $e^{j\pi n} = (-1)^n$. Thus, if

$$x_2[n] = (-1)^n x_1[n]$$

then

$$X_2(e^{j\omega}) = X_1(e^{j(\omega-\pi)}).$$

The figure below illustrates this with an example where $x_1[n]$ and $X_1(e^{j\omega})$ are shown at the top, and $x_2[n] = (-1)^n x_1[n]$ and $X_2(e^{j\omega})$ are at the bottom. Note that $X_1(e^{j\omega})$ is concentrated around $\omega = 0, \pm 2\pi, \cdots$ and $X_2(e^{j\omega})$ is concentrated around $\omega = \pm\pi, \pm 3\pi, \cdots$.



Example: Suppose a low-pass filter $H_{LP}(e^{j\omega})$ has been designed with impulse response $h_{LP}[n]$. To obtain a high-pass filter, let:

$$
\begin{aligned}
H_{HP}(e^{j\omega}) &= H_{LP}(e^{j(\omega-\pi)}) \\
h_{HP}[n] &= (-1)^n h_{LP}[n].
\end{aligned}
$$

Time Reversal:

$$\boxed{x[-n] \longleftrightarrow X(e^{-j\omega})} \tag{25}$$

Example:

$$
\begin{aligned}
a^n u[n] &\longleftrightarrow \frac{1}{1-ae^{-j\omega}} \\
a^{-n} u[-n] &\longleftrightarrow \frac{1}{1-ae^{j\omega}}
\end{aligned}
$$

Conjugation and Conjugate Symmetry:

$$\boxed{x^*[n] \longleftrightarrow X^*(e^{-j\omega})} \tag{26}$$

Thus, if $x[n]$ is real, then:

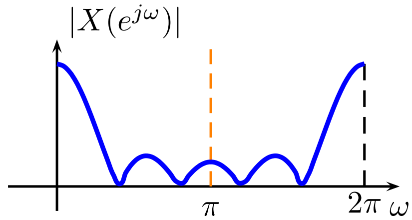
$$X(e^{j\omega}) = X^*(e^{-j\omega}) \tag{27}$$

or using the periodicity of $X(e^{j\omega})$:

$$X(e^{j\omega}) = X^*(e^{j(2\pi-\omega)}) \quad \Rightarrow \quad |X(e^{j\omega})| = |X(e^{j(2\pi-\omega)})| \tag{28}$$

$$\measuredangle X(e^{j\omega}) = -\measuredangle X(e^{j(2\pi-\omega)}) \tag{29}$$

Example 3 above:



Combining time reversal and conjugate symmetry:

$$x[-n] \longleftrightarrow X(e^{-j\omega}) = X^*(e^{j\omega}) \quad \text{(if } x[n] \text{ is real).} \tag{30}$$

Thus, if $x[n] = x[-n]$, then: $X(e^{j\omega}) = X^*(e^{j\omega})$, i.e., $X(e^{j\omega})$ is real. (See Examples 2 and 3 above.)

If $x[n] = -x[-n]$, then: $X(e^{j\omega}) = -X^*(e^{j\omega})$, i.e., $X(e^{j\omega})$ is purely imaginary.

Differencing:

$$\boxed{x[n] - x[n-1] \longleftrightarrow (1 - e^{-j\omega})X(e^{j\omega})} \tag{31}$$

Compare to CT: $\frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega)$

Accummulation:

$$\boxed{\sum_{m=-\infty}^{n} x[m] \longleftrightarrow \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)} \tag{32}$$

Compare to CT: $\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$

Example: $$\delta[n] \text{(unit impulse)} \longleftrightarrow \Delta(e^{j\omega}) \equiv 1 \tag{33}$$

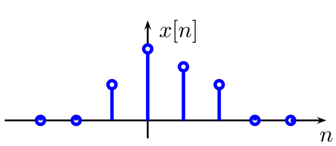
$$u[n] = \sum_{m=-\infty}^{n} \delta[m] \quad \longleftrightarrow \quad \frac{1}{1 - e^{-j\omega}} \underbrace{\Delta(e^{j\omega})}_{=1} + \pi \underbrace{\Delta(e^{j0})}_{=1} \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$
$$= \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \text{ [3]}$$

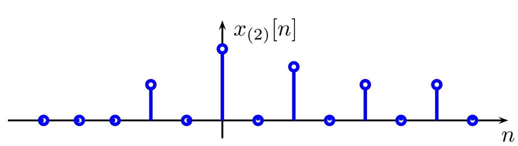
[3] Compare to Example 1; now $a = 1$.

Time Expansion:

Define

$$\boxed{x_{(M)}[n] = \begin{cases} x[n/M] & \text{if } n = 0, \mp M, \mp 2M, \ldots \\ 0 & \text{otherwise.} \end{cases}} \tag{34}$$





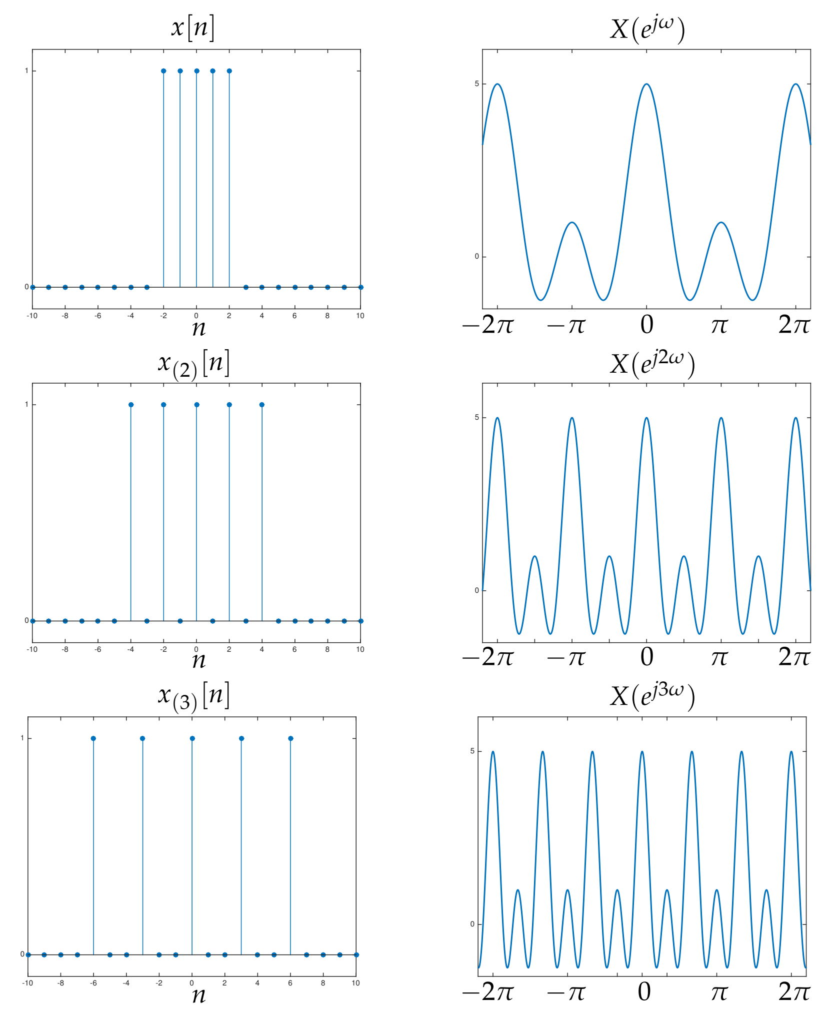
Then

$$\boxed{x_{(M)}[n] \longleftrightarrow X(e^{j\omega M}).} \tag{35}$$

Proof:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x_{(M)}[n]e^{-j\omega n} &= \sum_{k=-\infty}^{\infty} \underbrace{x_{(M)}[kM]}_{=x[k]} e^{-j\omega kM} \\ &= \sum_{k=-\infty}^{\infty} x[k]e^{-j(\omega M)k} = X(e^{j\omega M}). \end{aligned}$$

See figure below for an illustration on a rectangular pulse (top) expanded with $M = 2$ (middle) and $M = 3$ (bottom). The Fourier transforms shown on the right obey the property (35).



The time expansion property derived above is analogous to the continuous time property $x(at) \longleftrightarrow X(j\omega/a)$ with M playing the role of the scaling factor $1/a$.

Differentiation in Frequency:

$$\boxed{nx[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}} \tag{36}$$

Proof:

$$\begin{aligned} X(e^{j\omega}) &= \sum_n x[n]e^{-j\omega n} \\ \frac{dX(e^{j\omega})}{d\omega} &= -j\sum_n nx[n]e^{-j\omega n} \end{aligned}$$

Multiply both sides by j and substitute $-j^2 = 1$.

CT analog: $tx(t) \longleftrightarrow j\frac{dX(j\omega)}{d\omega}$

Parseval's Relation:

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega} \tag{37}$$