EE120 - Fall'15 - Lecture 5 Notes[1]

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Continuous Time Fourier Transform Continued

LTI Systems and the Convolution Property:

Recall from Lecture 2:

$$x(t) = e^{j\omega t} \rightarrow \boxed{h(t)} \rightarrow y(t) = H(j\omega)e^{j\omega t} \tag{1}$$

where

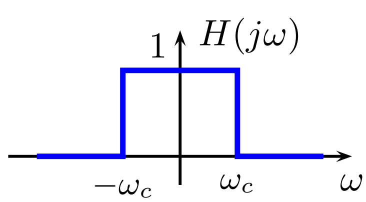
$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt \quad \text{"frequency response"}} \tag{2}$$

Thus, $H(j\omega)$ is the Fourier Transform of the impulse response $h(t)$. Note that the first Dirichlet condition for the existence of $H(j\omega)$:

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty \tag{3}$$

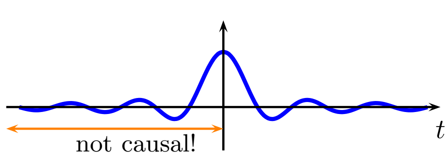
is equivalent to the stability of the system.

Question: Why is the ideal low pass filter "ideal"?



Answer: Synthesis equation applied to $H(j\omega)$ gives (see Lecture 4):

$$h(t) = \frac{sin\omega_c t}{\pi t} \tag{4}$$



The Convolution Property of the Fourier Transform

Section 4.4 *in* Oppenheim & Willsky

$$\boxed{h(t) * x(t) \overset{FT}{\longleftrightarrow} H(j\omega)X(j\omega)} \tag{5}$$

Proof:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$\begin{aligned} Y(j\omega) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(\tau)\underbrace{\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dtd\tau}_{=e^{-j\omega\tau}H(j\omega)^2} \end{aligned}$$

[2] follows from the time-shift property

$$\begin{aligned} &= H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \\ &= H(j\omega)X(j\omega) \end{aligned}$$

Example: $h(t) = e^{-at}u(t) \;\; a > 0 \quad \leftrightarrow \quad H(j\omega) = \frac{1}{a+j\omega}$

$x(t) = e^{-bt}u(t) \;\; b > 0 \quad \leftrightarrow \quad X(j\omega) = \frac{1}{b+j\omega}$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{(a+j\omega)(b+j\omega)} \tag{6}$$

Partial Fraction Expansion (if $a \neq b$):

$$Y(j\omega) = \frac{A}{a+j\omega} + \frac{B}{b+j\omega} = \frac{\overbrace{(Ab+Ba)}^{=1} + j\overbrace{(A+B)}^{=0\;3}\omega}{(a+j\omega)(b+j\omega)}$$

[3] From here: $B = -A$ and $Ab - Aa = 1$, which implies that

$$A = -B = \frac{1}{b-a}$$

$$\begin{aligned} Y(j\omega) &= \frac{1}{b-a}\left(\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right) \\ y(t) &= \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)u(t) \end{aligned}$$

If $a = b$:

$$Y(j\omega) = \frac{1}{(a+j\omega)^2} = j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right)$$

Use the property[4]: $\boxed{-jtx(t) \overset{FT}{\leftrightarrow} \frac{dX(j\omega)}{d\omega}}$

[4] dual of $\frac{dx(t)}{dt} \leftrightarrow j\omega X(j\omega)$

Then $y(t) = -j^2te^{-at}u(t) = te^{-at}u(t)$.

HW problem: Show that

$$\underbrace{e^{-at}u(t) * \ldots * e^{-at}u(t)}_{\text{r times}} = \frac{t^{r-1}}{(r-1)!}e^{-at}u(t)$$

The Frequency Shifting Property

Section 4.3 *in* Oppenheim & Willsky

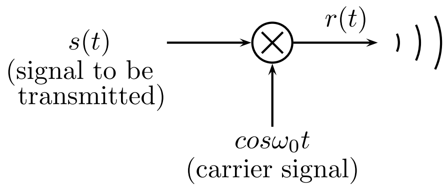
$$e^{j\omega_0 t}x(t) \stackrel{FT}{\longleftrightarrow} X(j(\omega-\omega_0)) \qquad (7)$$

dual of $x(t-t_0) \leftrightarrow e^{-j\omega_0 t}X(j\omega)$

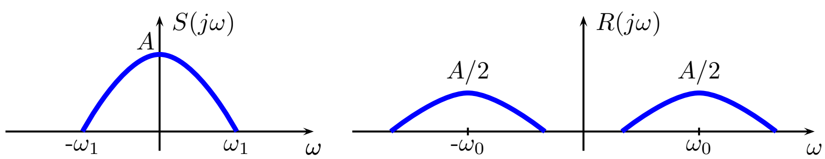
Proof:

$$\int_{-\infty}^{\infty} e^{j\omega_0 t}x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt = X(j(\omega-\omega_0))$$

Example: Amplitude Modulation (AM)

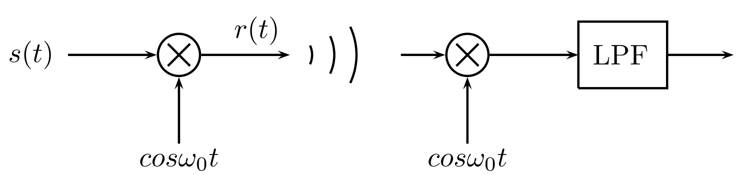


$$r(t) = s(t)\left(\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right) \longleftrightarrow R(j\omega) = \frac{1}{2}S(j(\omega-\omega_0)) + \frac{1}{2}S(j(\omega+\omega_0))$$

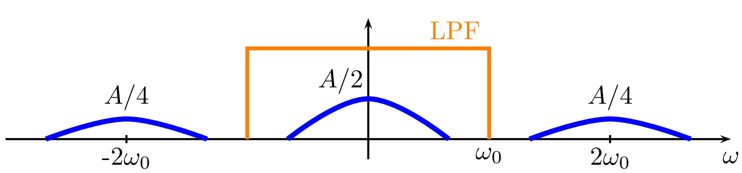


($\omega_0 >> \omega_1$ in practice)

Demodulation:



$$G(j\omega) = \frac{1}{2}R(j(\omega-\omega_0)) + \frac{1}{2}R(j(\omega+\omega_0))$$



The Multiplication Property

Section 4.5 *in* Oppenheim & Willsky

$$s(t)p(t) \longleftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega-\theta))d\theta \qquad (8)$$

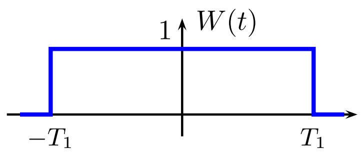
dual of the convolution property

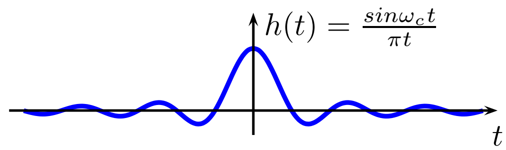
Proof: Apply the synthesis equation to the right-hand side above:

$$\begin{aligned} & \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega-\theta))d\theta\right)e^{j\omega t}d\omega \\ = & \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)\underbrace{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} P(j(\omega-\theta))e^{j\omega t}d\omega\right)}_{=e^{j\theta t}p(t)\ 5} d\theta \\ & = p(t)\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)e^{j\theta t}d\theta}_{=s(t)} \end{aligned}$$

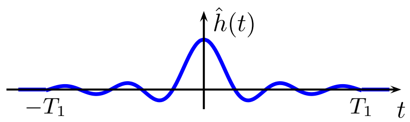
[5] from the frequency shift property

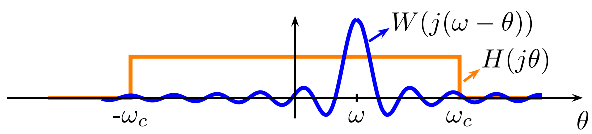
Example: Truncating the impulse response of the ideal low-pass filter



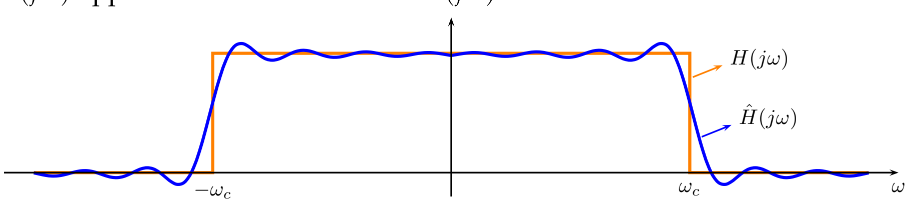


$$\hat{h}(t) \triangleq w(t)\cdot h(t) \quad \overset{FT}{\leftrightarrow} \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\theta)W(j(\omega-\theta))d\theta$$



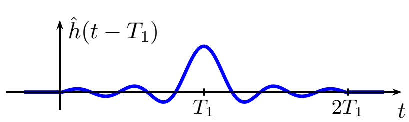


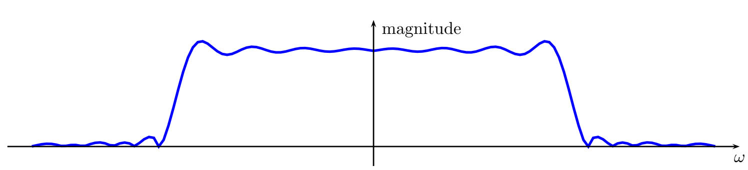
$\hat{H}(j\omega)$ approximates the ideal filter $H(j\omega)$:



but it is still non-causal. Causal version:

$$\hat{h}(t-T_1) \quad \overset{FT}{\leftrightarrow} \quad e^{-j\omega T_1}\hat{H}(j\omega)$$





Same magnitude as $\hat{H}(j\omega)$:

$$\left|e^{-j\omega T_1}\hat{H}(j\omega)\right| = |\hat{H}(j\omega)|$$

but $e^{-j\omega T_1}\hat{H}(j\omega)$ has an additional phase of $-\omega T_1$ due to the delay.

Frequency Response of Continuous Time LTI Systems

Sections 4.7 and 6.5 *in* Oppenheim & Willsky

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{9}$$

Take Fourier transforms of both sides and apply the differentiation property:

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega) \tag{10}$$

Then,

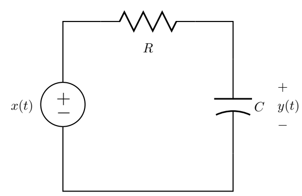
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}. \tag{11}$$

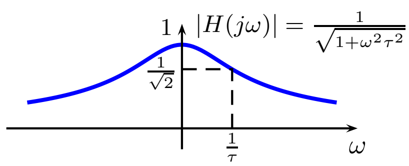
Example: First-order system:

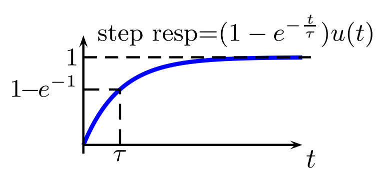
$$\tau \frac{dy}{dt} + y(t) = x(t) \tag{12}$$

where $\tau > 0$: "time constant" (e.g., $\tau = RC$ in the RC circuit)

$$H(j\omega) = \frac{1}{1 + j\omega\tau}$$







Example: Second-order system:

$$\frac{d^2 y}{dt^2} + 2\xi\omega_n \frac{dy}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t) \tag{13}$$

where ξ is called the damping ratio, and ω_n the natural frequency.

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2} = \frac{1}{(j\omega/\omega_n)^2 + 2\xi(j\omega/\omega_n) + 1}$$

The figure below shows the frequency, impulse, and step responses for various values of ξ. Note that increasing ω_n stretches the frequency response along the ω axis and compresses the impulse and step responses along the t axis. Therefore, a large natural frequency means faster response.

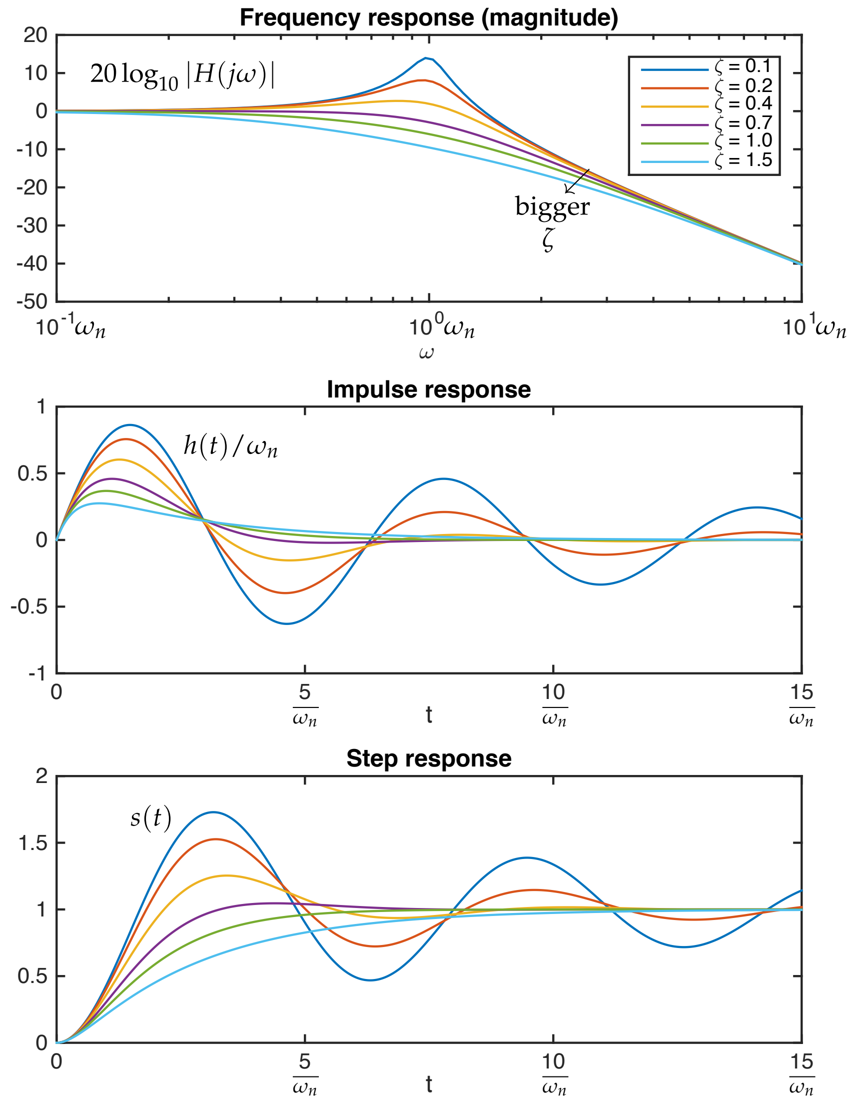


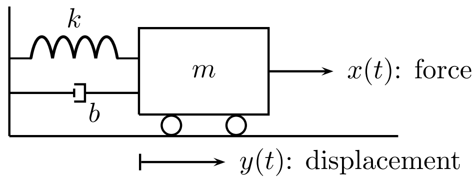
Figure 1: The frequency, impulse, and step responses for the second order system (13). Note from the frequency response (top) that a resonance peak occurs when $\zeta < 0.7$.

When does resonance occur?

$$|H(j\omega)|^2 = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_n^2}} = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^4 + (4\xi^2 - 2)\left(\frac{\omega}{\omega_n}\right)^2 + 1}$$

Note that the denominator is strictly increasing in ω if $4\xi^2 - 2 \geq 0$ and has a minimum at some $\omega > 0$ otherwise. Thus, if $4\xi^2 - 2 < 0$ (i.e., $\xi < 1/\sqrt{2} \approx 0.7$), then $|H(j\omega)|$ has a resonance peak as confirmed with the frequency response shown in the top figure above.

Example:



$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = x$$
$$\frac{d^2y}{dt^2} + \underbrace{\left(\frac{b}{m}\right)}_{2\xi\omega_n}\frac{dy}{dt} + \underbrace{\left(\frac{k}{m}\right)}_{\omega_n^2} y = \frac{1}{m}x$$

with $\xi = \frac{b}{2\sqrt{km}}$ and $\omega_n = \sqrt{\frac{k}{m}}$. Ressonance occurs if $\xi^2 = \frac{b^2}{4km} < \frac{1}{2}$, i.e., if $b^2 < 2km$.