EE120 - Fall'15 - Lecture 4 Notes[1]

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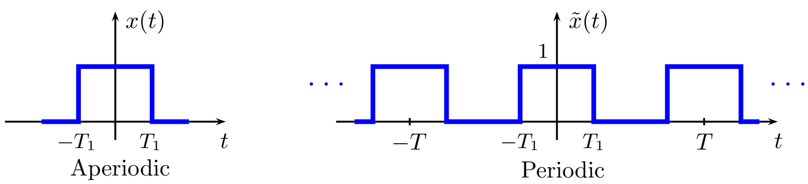
Continuous Time Fourier Transform

Chapter 4 *in* Oppenheim & Willsky

Applicable to aperiodic signals (unlike Fourier series which is applicable only to periodic signals).
Main idea: Treat aperiodic signal $x(t)$ as the limit of a periodic signal $\tilde{x}(t)$ as period $T \to \infty$ (see figure below). As T increases, the fundamental frequency $\omega_0 = \frac{2\pi}{T}$ decreases and the harmonic components become closer in frequency, forming a continuum in the limit $T \to \infty$.

Example:



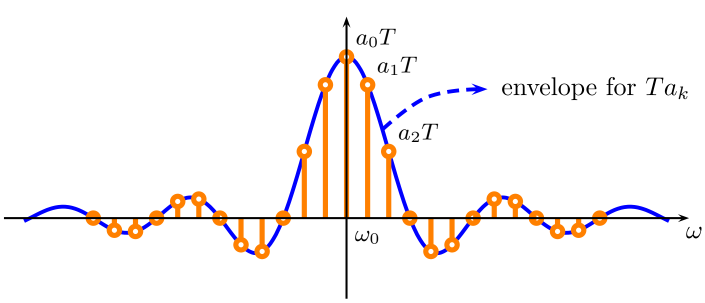
Recall from Lecture 3 that the periodic signal on the right has Fourier series coefficients:

$$a_k = \frac{2sin(k\omega_0 T_1)}{k\omega_0 T}, \quad \omega_0 = \frac{2\pi}{T}. \tag{1}$$

This expression is not defined for $k = 0$, but we interpret a_0 to be the limit as $k \to 0$, *i.e.*, $a_0 = 2T_1/T$.

Then

$$Ta_k = \frac{2sin(\omega T_1)}{\omega}\bigg|_{\omega = k\omega_0} \tag{2}$$



This envelope is the "Fourier transform" of $x(t)$ above. In general:

$$a_k = \frac{1}{T}\int_T \tilde{x}(t)e^{-jk\omega_0 t}dt \tag{3}$$

$$Ta_k = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt = \underbrace{\left(\int_{\infty}^{\infty} x(t)e^{-j\omega t}dt\right)}_{\triangleq X(j\omega) \text{ (Fourier Transform)}}\Bigg|_{\omega = k\omega_0} \tag{4}$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt && \text{(Analysis Equation)} \\ x(t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega && \text{(Synthesis Equation)} \end{aligned} \tag{5}$$

Derivation of the synthesis equation:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \underbrace{\frac{2\pi}{T}}_{=\omega_0} \left[X(j\omega)e^{j\omega t}\right]\Big|_{\omega=k\omega_0} \text{ [2]} \tag{6}$$

Take the limit as $T \to \infty$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \tag{7}$$

Convergence: If the "Dirichlet conditions" below hold, then

$$\tfrac{1}{2\pi}\int_{-W}^{W} X(j\omega)e^{j\omega t}d\omega$$

converges to $x(t)$ as $W \to \infty$ for all t, except at discontinuities where it converges to the average:
D1) $x(t)$ is absolutely integrable: $\int_{\infty}^{\infty} |x(t)|dt < \infty$;
D2) $x(t)$ has finite # of minima and maxima within any finite interval;
D3) $x(t)$ has finite # of discontinuities within any finite interval and the discontinuities are finite.

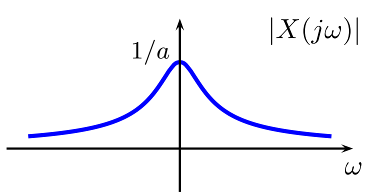
Examples:

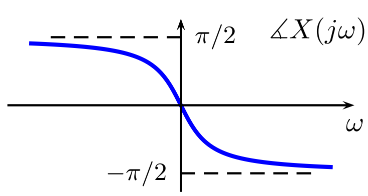
1) $x(t) = e^{-at}u(t),\ a > 0.$

$$X(j\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = \int_0^{\infty} e^{-(a+j\omega)t}dt = \frac{1}{a+j\omega}\underbrace{e^{-(a+j\omega)t}\Big|_0^{\infty}}_{=1}$$

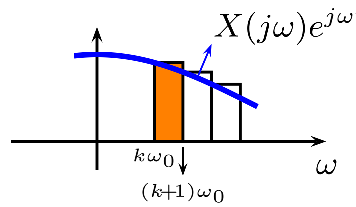
$$X(j\omega) = \frac{1}{a+j\omega}, \quad |a+j\omega| = \sqrt{a^2+\omega^2}, \quad \measuredangle a+j\omega = tan^{-1}(\omega/a)$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \quad \measuredangle X(j\omega) = -\tan^{-1}(\omega/a)$$





[2] The summation term on the right can be pictured as:

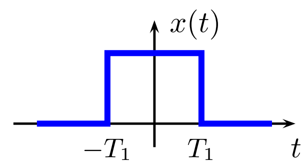


2) Dirac delta:

$$\begin{aligned} x(t) &= \delta(t) \\ X(j\omega) &= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1 \ \text{ for all } \omega \end{aligned}$$

3) Rectangular pulse:

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| \geq T_1 \end{cases}$$



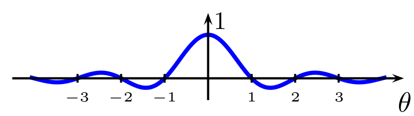
$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt$$

For $\omega = 0$, $X(j0) = \int_{-T_1}^{T_1} dt = 2T_1$. For $\omega \neq 0$,

$$\int_{-T_1}^{T_1} e^{-j\omega t} dt = -\frac{1}{j\omega} e^{-j\omega t}\Big|_{T_1}^{T_1} = \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega} = \frac{2 sin(\omega T_1)}{\omega}.$$

Combining,

$$sinc(\theta) \triangleq \begin{cases} \frac{sin \pi\theta}{\pi\theta} & \theta \neq 0 \\ 1 & \theta = 0. \end{cases}$$



$$X(j\omega) = \begin{cases} 2T_1 & \omega = 0 \\ \frac{2sin(\omega T_1)}{\omega} & \omega \neq 0 \end{cases} = 2T_1 \text{sinc}\left(\frac{T_1}{\pi}\omega\right)$$

4)

$$X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| \geq W \end{cases}$$

A derivation similar to Example 3 gives:

$$x(t) = \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t} d\omega = \frac{W}{\pi} sinc\left(\frac{W}{\pi} t\right)$$

Note the duality in Examples 3 and 4.

rectangular pulse	$\overset{FT}{\leftrightarrow}$	sinc
sinc	$\overset{FT}{\leftrightarrow}$	rectangular pulse

5)

$$\begin{aligned} X(j\omega) &= 2\pi\delta(\omega - \omega_0) \\ x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t} \end{aligned}$$

If $\omega_0 = 0$, then $1 \overset{FT}{\leftrightarrow} 2\pi\delta(\omega)$. Note the duality with Example 2.

Fourier Transform of Periodic Signals

Section 4.2 *in* Oppenheim & Willsky

From Example 5 above:

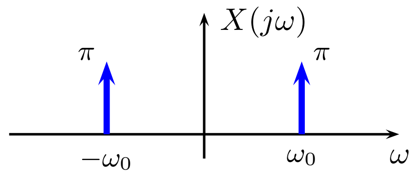
$$e^{j\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi\delta(\omega - \omega_0) \tag{8}$$

By linearity:

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \overset{FT}{\leftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \tag{9}$$

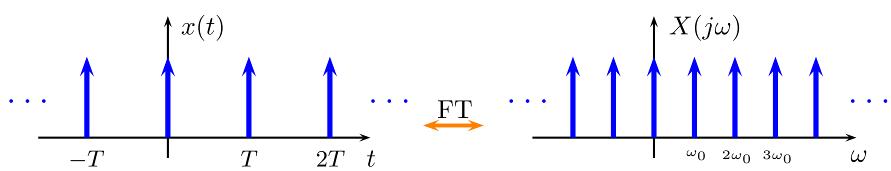
Example:

$$x(t) = cos(\omega_0 t) = \tfrac{1}{2}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\omega_0 t}$$
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Example: Impulse Train

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} \delta(t - kT) \\ a_k &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 t}dt = \frac{1}{T} \quad \text{for all } k \\ X(j\omega) &= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \end{aligned}$$



Properties of the Fourier Transform

Section 4.3 *in* Oppenheim & Willsky

Consider $x(t) \overset{FT}{\leftrightarrow} X(j\omega)$ and $y(t) \overset{FT}{\leftrightarrow} Y(j\omega)$.

Linearity:

$$\boxed{ax(t) + by(t) \overset{FT}{\leftrightarrow} aX(j\omega) + bY(j\omega), \quad a, b \in \mathcal{R}} \tag{10}$$

Time-Shift:

$$\boxed{x(t - t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0}X(j\omega)} \tag{11}$$

Proof:

$$\int_{-\infty}^{\infty} x(\underbrace{t-t_0}_{\triangleq \tau})e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(t_0+\tau)}d\tau$$
$$= e^{-j\omega t_0}\underbrace{\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau}_{=X(j\omega)}$$

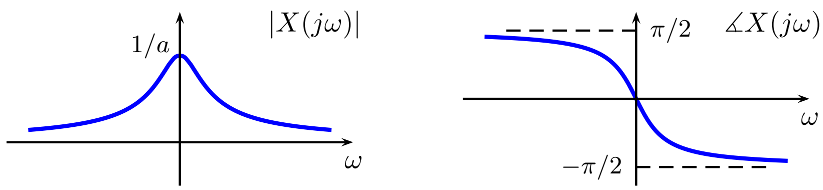
Conjugation and Conjugate Symmetry

$$\boxed{x^*(t) \overset{FT}{\leftrightarrow} X^*(-j\omega)} \tag{12}$$

If $x(t)$ is real: $X(j\omega) = X^*(-j\omega)$ (because $x(t) = x^*(t)$)

$$\Rightarrow |X(j\omega)| = |X(-j\omega)| \quad \text{(even symmetry)} \tag{13}$$
$$\measuredangle X(j\omega) = -\measuredangle X(-j\omega) \quad \text{(odd symmetry)} \tag{14}$$

Example 1 above:



Differentiation and Integration

$$\boxed{\frac{dx(t)}{dt} \overset{FT}{\leftrightarrow} j\omega X(j\omega)} \tag{15}$$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \overset{FT}{\leftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)} \tag{16}$$

Example: $x(t) = u(t)$ (unit step)

Note $x(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$ and $\delta(t) \overset{FT}{\leftrightarrow} \Delta(j\omega) \equiv 1$.

Thus, $X(j\omega) = \frac{1}{j\omega}\Delta(j\omega) + \pi\Delta(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$

Time and Frequency Scaling

$$\boxed{x(at) \overset{FT}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{j\omega}{a}\right), \quad a \neq 0} \tag{17}$$

Proof:

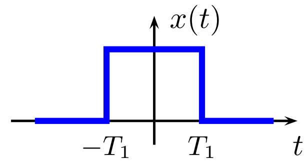
$$\int_{-\infty}^{\infty} x(\underbrace{at}_{\triangleq\tau})e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau/a}\frac{d\tau}{a}, \text{ if } a>0$$

$$= \int_{\infty}^{-\infty} x(\tau)e^{-j\omega\tau/a}\frac{d\tau}{a}, \text{ if } a<0$$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau/a}d\tau = \frac{1}{|a|}X\left(j\frac{\omega}{a}\right)$$

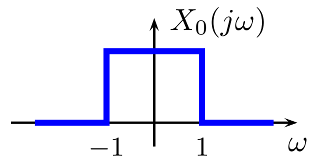
Example:

$$x(t) = \frac{sinWt}{\pi t} \quad \overset{\textbf{FT}}{\longleftrightarrow}$$



We can interpret this as a scaling of:

$$x_0(t) = \frac{sint}{\pi t} \quad \overset{\textbf{FT}}{\longleftrightarrow}$$



$$x(t) = W\frac{sinWt}{\pi Wt} = Wx_0(Wt) \overset{\textbf{FT}}{\longleftrightarrow} X_0(j\omega/W) = X(j\omega)$$

Corollary (a=-1): $x(-t) \leftrightarrow X(-j\omega)$ (18)

If $x(-t) = x(t)$ then $X(-j\omega) = X(j\omega)$
If $x(t)$ is also real: $X(-j\omega) = X^*(j\omega)$
} $X(j\omega) = X^*(j\omega)$, i.e., $X(j\omega)$is real.

Parseval's Relation:

$$\boxed{\int_{-\infty}^{\infty}|x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2d\omega} \tag{19}$$

Example: $x(t) = e^{-at}u(t) \;\; a>0 \leftrightarrow X(j\omega) = \dfrac{1}{a+j\omega}$

$$\int_{-\infty}^{\infty}|x(t)|^2dt = \int_0^{\infty}e^{-2at}dt = \frac{1}{2a}\,e^{-2at}\Big|_{\infty}^{0} = \frac{1}{2a}$$

$$\int_{-\infty}^{\infty}|X(j\omega)|^2d\omega = \int_{-\infty}^{\infty}\frac{1}{a^2+\omega^2}d\omega = \frac{1}{a}\,tan^{-1}\left(\frac{\omega}{a}\right)\Big|_{-\infty}^{\infty} = \frac{\pi}{a} = 2\pi\frac{1}{2a}$$

Initial Value:

$$\boxed{x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)d\omega \quad \text{(synthesis eq'n with } t=0)} \tag{20}$$

DC Component:

$$\boxed{X(0) = \int_{-\infty}^{\infty}x(t)dt \quad \text{(analysis equation with } \omega=0)} \tag{21}$$