EE120 - Fall'15 - Lecture 3 Notes[1]

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Fourier Series for Continuous-Time Periodic Signals

Section 3.3 *in* Oppenheim & Willsky

Recall:

$$\left.\begin{array}{rcl} e^{j\theta} & = & \cos\theta + j\sin\theta \\ (e^{j\theta})^* & = & e^{-j\theta} = \cos\theta - j\sin\theta \end{array}\right\} \quad \begin{array}{rcl} \cos\theta & = & \frac{1}{2}\left(e^{j\theta}+e^{-j\theta}\right) \\ \sin\theta & = & \frac{1}{2j}\left(e^{j\theta}-e^{-j\theta}\right) \end{array}$$

and $e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$ is a periodic signal with period $T = \frac{2\pi}{\omega_0}$.

Fourier Series represents a periodic signal $x(t+T) = x(t)\ \forall t$ as a weighted sum of sinusoidals $e^{jk\omega_0 t}$ $k = 0, \mp 1, \mp 2, \ldots$

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \omega_0 = \frac{2\pi}{T} \quad \text{(synthesis equation)}} \tag{1}$$

$k = 0$: $a_k e^{jk\omega_0 t} \equiv a_0$ ("dc component")
$k = \mp 1$: fundamental frequency ("first harmonic")
$k = \mp 2$: "second harmonic"

Example: $$x(t) = 1 + \underbrace{\frac{1}{2}\cos(2\pi t)}_{=\frac{1}{4}e^{j2\pi t}+\frac{1}{4}e^{-j2\pi t}} + \underbrace{\sin(4\pi t)}_{=\frac{1}{2j}e^{j4\pi t}-\frac{1}{2j}e^{-j4\pi t}} + \underbrace{\frac{2}{3}\cos(6\pi t)}_{=\frac{1}{3}e^{j6\pi t}+\frac{1}{3}e^{-j6\pi t}} \tag{2}$$

then $a_0 = 1$, $a_1 = a_{-1} = \frac{1}{4}$, $a_2 = -a_{-2} = \frac{1}{2j}$, $a_3 = a_{-3} = \frac{1}{3}$.

Property: For a real signal $x(t) = x^*(t)$, $a_k = a^*_{-k}$.

Proof: Follows from the "conjugate symmetry" property: If $x(t)$ has Fourier series coefficient a_k, then $x^*(t)$ has Fourier series coefficients $b_k = a^*_{-k}$. If $x(t)$ is real, then $x(t) = x^*(t)$; therefore, $a_k = b_k = a^*_{-k}$.

How to find the Fourier Series coefficients a_k?

Multiply both sides of the synthesis equation (1) with $e^{-jn\omega_0 t}$ and integrate from 0 to $T = \frac{2\pi}{\omega_0}$:

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \underbrace{\sum_{k=-\infty}^{\infty} a_k \underbrace{\left(\int_0^T e^{j(k-n)\omega_0 t}dt\right)}_{=\begin{cases} T & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}}}_{=Ta_n} \tag{3}$$

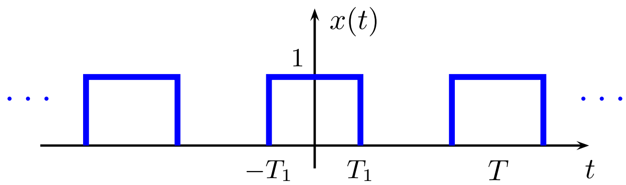
Jean-Baptiste-Joseph Fourier
(1768-1830)

Therefore:

$$\boxed{a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt \quad \text{(analysis equation)}} \tag{4}$$

In particular, $a_0 = \frac{1}{T}\int_0^T x(t)dt$ (average of $x(t)$ over one period).

Example: Periodic Square Wave



For $k = 0$,

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T} \tag{5}$$

For $k \neq 0$,

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = \frac{1}{T}\frac{-1}{jk\omega_0}\underbrace{e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}}_{\substack{=e^{jk\omega_0 T_1}-e^{-jk\omega_0 T_1} \\ =-2jsin(k\omega_0 T_1)}} \tag{6}$$

$$= \frac{2}{k\omega_0 T}\sin(k\omega_0 T_1) = \frac{1}{k\pi}\sin(2\pi k\tfrac{T_1}{T}) \tag{7}$$

Discrete-Time Periodic Signals

A discrete-time signal $x[n]$ is periodic if there exists integer $N \neq 0$ s.t.

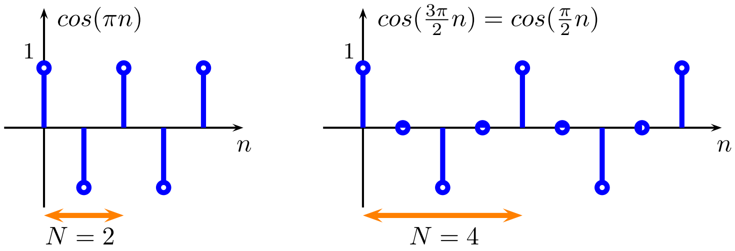
$$x[n+N] = x[n] \quad \text{for all } n. \tag{8}$$

Question: Is $x[n] = cos(\omega_0 n)$ periodic for any ω_0?
Answer: No. It is periodic only when ω_0/π is rational. To find the fundamental period N, find the smallest integers M, N such that

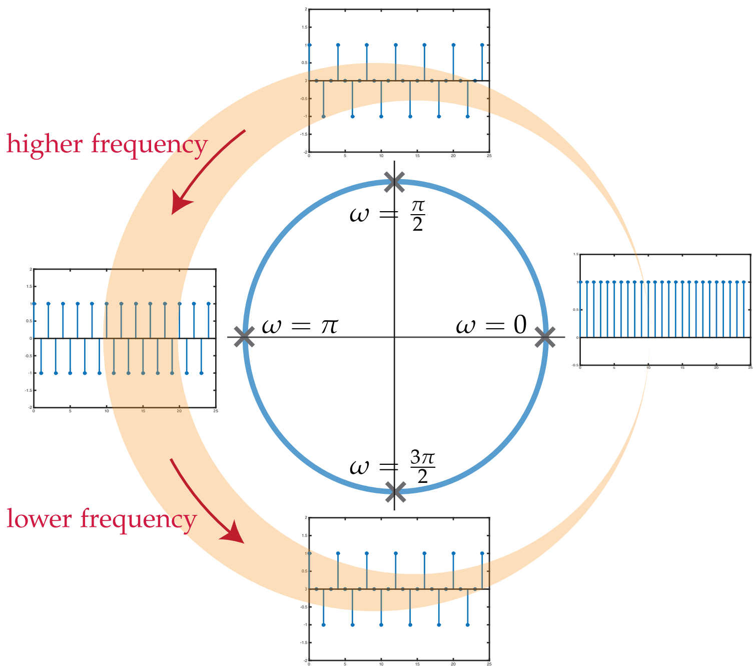
$$\boxed{\omega_0 N = 2\pi M} \tag{9}$$

Examples:
1. $\cos(n)$ is not periodic;
2. $\cos(\frac{5\pi}{7}n)$, $N = 14$;
3. $\cos(\frac{\pi}{5}n)$, $N = 10$;
4. $\cos(\frac{5\pi}{7}n) + cos(\frac{\pi}{5}n)$, $N = s.c.m.\{14, 10\} = 70$.

Question: Which one is a higher frequency, $\omega_0 = \pi$ or $\omega_0 = \frac{3\pi}{2}$?
Answer: $\omega_0 = \pi$



In discrete time $\omega = \pi$ is the highest frequency, as depicted below.



Discrete-Time Fourier Series

The complex exponential signal

$$e^{j\omega_0 n} = cos(\omega_0 n) + jsin(\omega_0 n)$$

is periodic if $\omega_0 N = 2\pi M$ for some integers M, N:

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n} \underbrace{e^{j\omega_0 N}}_{=e^{j2\pi M}=1} = e^{j\omega_0 n} \tag{10}$$

The Fourier Series expresses the periodic sequence $x[n + N] = x[n]$ as a linear combination of

$$\boxed{\Phi_k[n] \triangleq e^{jk\omega_0 n}, \quad k = 0, \mp 1, \mp 2, ..., \quad \omega_0 = \frac{2\pi}{N}} \tag{11}$$

Key difference between CT and DT:

$$e^{j(k+N)\omega_0 n} = e^{jk\omega_0 n} \tag{12}$$

because $e^{jN\omega_0 n} = e^{j2\pi M n} = 1$. Therefore,

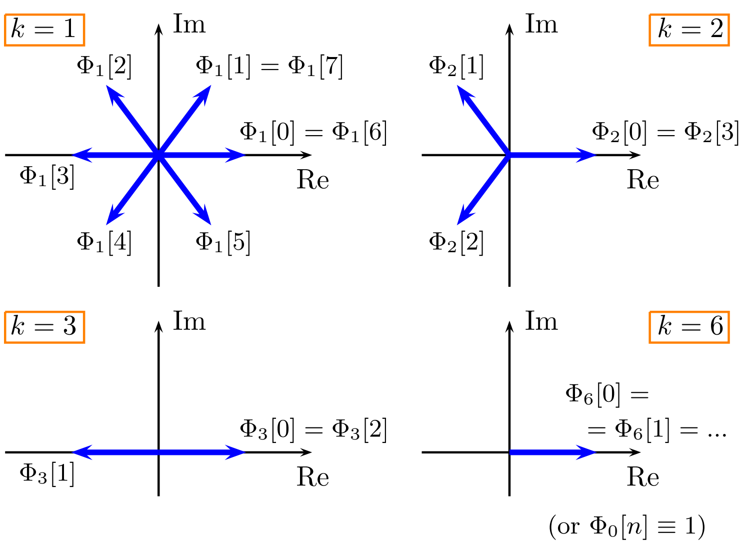
$$\Phi_k[n] = \Phi_{k+N}[n] = \Phi_{k+2N}[n] = ... \tag{13}$$

and N independent functions $\Phi_k[n]$ (*e.g.*, $\Phi_0[n], \Phi_1[n], ..., \Phi_{N-1}[n]$) are enough for the Fourier Series. We use the finite series

$$\boxed{x[n] = \sum_{k=\langle N \rangle} a_k \Phi_k[n] \quad \text{(Synthesis Equation)}} \tag{14}$$

where $k = \langle N \rangle$ means any set of N successive integers: $k = 0, 1, ..., N-1$, or $k = 1, 2, ..., N$, or other choices.

Example: For $N = 6$, $\Phi_k[n] = e^{jk\frac{2\pi}{6}n}$



Properties of $\Phi_k[n]$:

1. Periodicity in n: $\Phi_k[n+N] = \Phi_k[n]$;
2. Periodicity in k: $\Phi_{k+N}[n] = \Phi_k[n]$;
3. $$\sum_{n=\langle N \rangle} \Phi_k[n] = \begin{cases} N & \text{if } k = 0, \mp N, \mp 2N, ... \\ 0 & \text{otherwise} \end{cases} \tag{15}$$
4. $\Phi_k[n] \cdot \Phi_m[n] = \Phi_{k+m}[n]$ (follows from definition $\Phi_k[n] = e^{jk\frac{2\pi}{N}n}$)

Finding the Fourier Series coefficients a_k:

Multiply both sides of (14) by $\Phi_{-m}[n]$ and sum over $n = \langle N \rangle$:

$$\begin{aligned}
\sum_{n=\langle N\rangle} x[n]\Phi_{-m}[n] &= \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k\Phi_{k-m}[n] && (16)\\
&= \sum_{k=\langle N\rangle} a_k \underbrace{\sum_{n=\langle N\rangle}\Phi_{k-m}[n]}_{=\begin{cases} N & \text{if } k=m(mod N)\\ 0 & \text{otherwise}\end{cases}} && (17)\\
&= Na_m && (18)
\end{aligned}$$

Replace $m \to k$:

$$\boxed{a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-j\frac{2\pi}{N}kn} \quad \text{(Analysis Equation)}} \qquad (19)$$

Summary:

	CT	DT
Synthesis	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$
Analysis	$a_k = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{T}t}$	$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\frac{2\pi}{N}n}$

Example:

$$\begin{aligned}
x[n] &= 1 + \underbrace{\sin\left(\frac{2\pi}{10}n\right)}_{=\frac{1}{2j}e^{j\frac{2\pi}{10}n} - \frac{1}{2j}e^{-j\frac{2\pi}{10}n}} + \underbrace{\cos\left(\frac{4\pi}{10}n + \frac{\pi}{4}\right)}_{=\frac{1}{2}e^{j\frac{\pi}{4}}e^{j\frac{4\pi}{10}n} + \frac{1}{2}e^{-j\frac{\pi}{4}}e^{-j\frac{4\pi}{10}n}} \qquad N = 10\\
&= 1 + \frac{1}{2j}\Phi_1[n] - \frac{1}{2j}\Phi_{-1}[n] + \frac{1}{2}e^{j\frac{\pi}{4}}\Phi_2[n] + \frac{1}{2}e^{-j\frac{\pi}{4}}\Phi_{-2}[n]
\end{aligned}$$

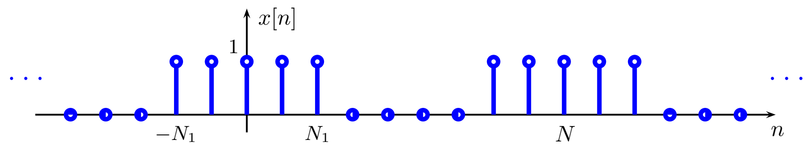
If we choose $\langle N \rangle$ to be $\{0, 1, 2, ..., 9\}$, then

$$a_0 = 1,\ a_1 = \tfrac{1}{2j},\ a_2 = \tfrac{1}{2}e^{j\frac{\pi}{4}},\ a_3 = a_4 = a_5 = a_6 = a_7 = 0, \qquad (20)$$

$$a_8 = \tfrac{1}{2}e^{-j\frac{\pi}{4}},\ a_9 = -\tfrac{1}{2j} \qquad (21)$$

Note: As in CT, $x[n]$ real implies that $a_{-k} = a_k^*$. Combined with the periodicity of coefficients in DT ($a_{N-k} = a_{-k}$): $\boxed{a_{N-k} = a_k^*}$.

<u>Example:</u> Rectangular pulse train



For the special case $N_1 = 0$ ("impulse train"):

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} x[0] e^{-jk\frac{2\pi}{N}0} = \frac{1}{N} \quad \forall k.$$

Derive the following for $N_1 \neq 0$:

$$a_k = \begin{cases} \frac{2N_1+1}{N} & k = 0 \\ \frac{1}{N} \frac{\sin(k\pi(2N_1+1)/N)}{\sin(k\pi/N)} & k \neq 0. \end{cases} \tag{22}$$

The figure below shows how the partial sum

$$\sum_{k=-M}^{M} a_k \Phi_k[n] \tag{23}$$

progressively reconstructs $x[n]$ as more harmonics are included.

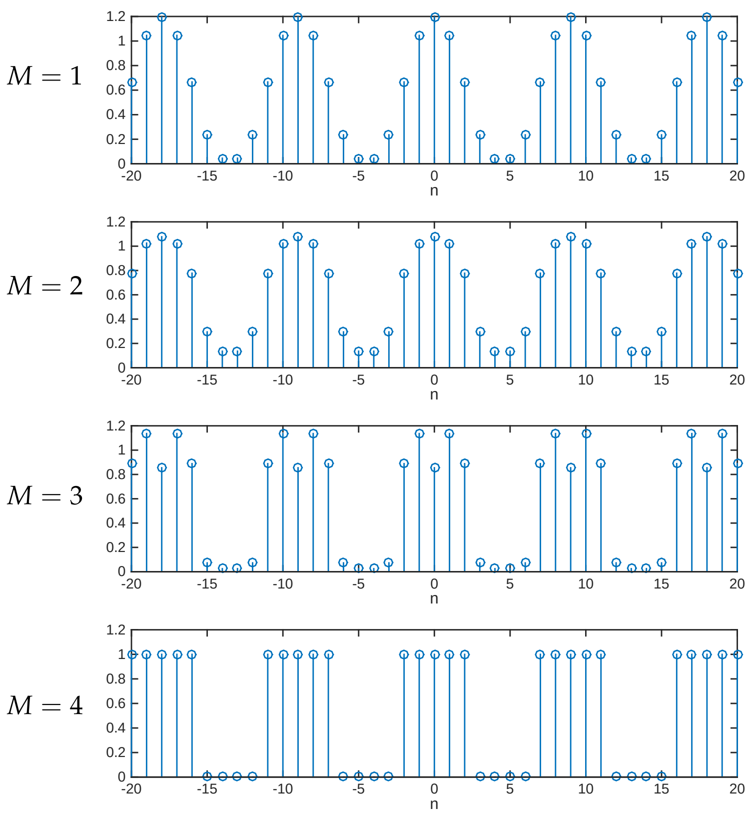


Figure 1: The partial sum (23) with Fourier coefficients (22), for $N = 9$ and $N_1 = 2$. When $M = 4$, (23) is the complete Fourier series; thus we fully recover the rectangular pulse.

Fourier Series as a "Change of Basis"

Consider the period-two signal:

$$x[n] = \begin{cases} 2 & \text{if n even} \\ 3 & \text{if n odd.} \end{cases}$$

We have $N = 2$ and the Fourier series is:

$$x[n] = a_0\Phi_0[n] + a_1\Phi_1[n]$$

where $\Phi_0[n] \equiv 1$, $\Phi_1[n] = (-1)^n$. Applying the analysis equation, you can show that:

$$a_0 = \frac{5}{2} \quad a_1 = -\frac{1}{2}.$$

Now view $x[n]$ as a vector:

$$x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

whose entries are the values $x[n]$ takes at $n = 0, 1$ and the dimension is two because the period is $N = 2$.

Then,

$$\Phi_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Phi_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

can be viewed as new basis vectors and the Fourier series can be interpreted as a change of basis:

$$x = a_0\Phi_0 + a_1\Phi_1 = \frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

The advantage of the new basis is that, instead of the values in time, the signal is represented with coefficients of its frequency components. This allows, for example, compression algorithms that allocate more bits to accurately store the coefficients of frequency components that matter more to the quality of sound than other frequencies.