EE120 - Fall'15 - Lecture 22 Notes[1]

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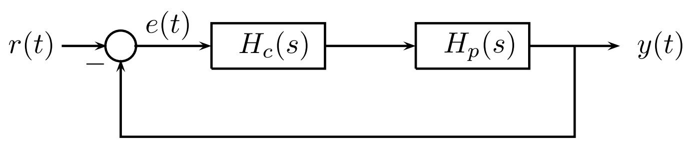
Steady State Accurary



$$
\begin{aligned}
e(t) &= r(t) - y(t) \\
E(s) &= R(s) - Y(s) \\
&= R(s) - \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)} R(s) = \frac{1}{1 + H_c(s)H_p(s)} R(s)
\end{aligned}
$$

Suppose $r(t)$ is a unit step. How do we guarantee $e(t)$ converges to zero instead of a different constant?

$$
R(s) = \frac{1}{s} \implies E(s) = \frac{1}{1 + H_c(s)H_p(s)} \cdot \frac{1}{s}
$$

Final Value Theorem:

$$
e_{ss} \triangleq \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \frac{1}{1 + H_c(0)H_p(0)}
$$

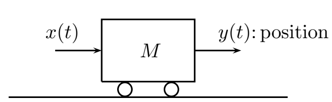
To ensure $e_{ss} = 0$, we need $\lim_{s \to 0} H_c(s)H_p(s) = \infty$, *i.e.*,

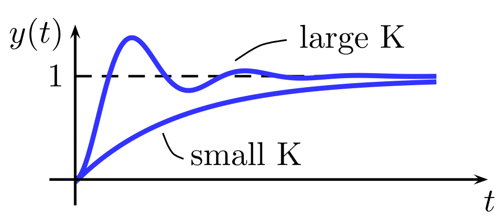
> $H_c(s)H_p(s)$ must have at least one pole at $s = 0$.

Example: Position control

$$
H_p(s) = \frac{1}{Ms^2 + bs} \qquad H_c(s) = K
$$

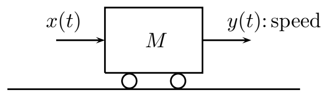
$$
H_c(s)H_p(s) = \frac{K}{s(Ms + b)} \rightarrow e_{ss} = 0
$$



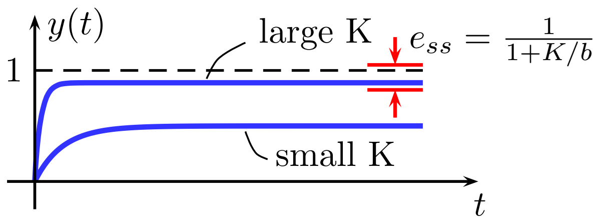


Example: Speed control

$$H_p(s) = \frac{1}{Ms+b} \quad H_c(s) = K$$



$$e_{ss} = \frac{1}{1+H_c(0)H_p(0)} = \frac{1}{1+K/b} \neq 0$$
$$y_{ss} = 1 - e_{ss} = 1 - \frac{1}{1+K/b} = \frac{K/b}{1+K/b}$$



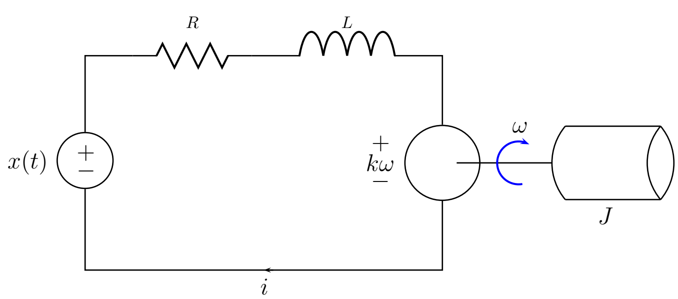
Steady-state error decreases with increasing K, but increasing K is not always a viable approach (poor damping if #poles − #zeros = 2, instability if #poles − #zeros ≥ 3).

Integral Control

If $H_p(s)$ does not contain a pole at $s = 0$, introduce one in $H_c(s)$.

Drawback: pole at $s = 0$ makes it harder to meet damping and natural frequency specifications.

Example: Speed control of a DC motor



Suppose we want to control $y(t) = \omega$ (angular velocity).

First, find the transfer function $H_p(s)$:

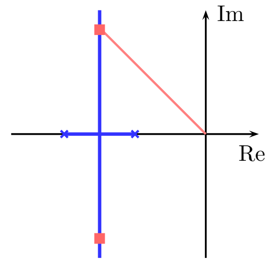
$$J\frac{d\omega(t)}{dt} = ki(t)$$
$$L\frac{di(t)}{dt} = -k\omega(t) - Ri(t) + x(t)$$

Take the Laplace transform and substitute $y = \omega$:

$$\begin{aligned} JsY(s) &= kI(s) \\ LsI(s) &= -kY(s) - RI(s) + X(s) \end{aligned}$$

Substitute $I(s) = \frac{X(s)-kY(s)}{Ls+R}$ from the second equation into the first:

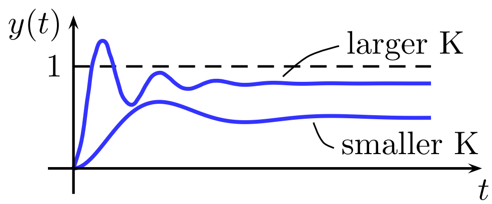
$$JsY(s) = k\frac{X(s)-kY(s)}{Ls+R}$$
$$[Js(Ls+R)+k^2]Y(s) = kX(s)$$
$$H_p(s) = \frac{Y(s)}{X(s)} = \frac{k}{JLs^2+JRs+k^2}$$



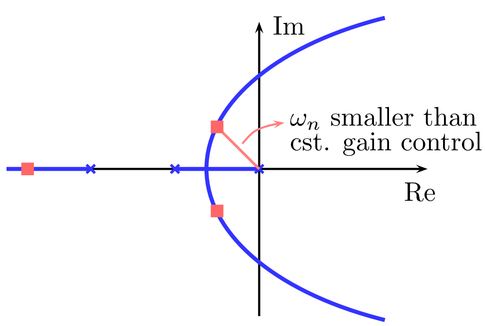
Constant gain control $H_c(s) = K$ gives nonzero steady-state error:

$$e_{ss} = \frac{1}{1+KH_p(0)} = \frac{1}{1+\frac{K}{k}} \neq 0$$

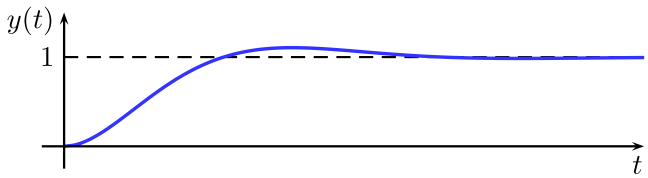
Increasing the gain reduces e_{ss}, but leads to a poorly damped system:



Integral Control: $H_c(s) = \frac{K}{s}$



$e_{ss} = 0$ achieved at the cost of slower response (smaller ω_n):

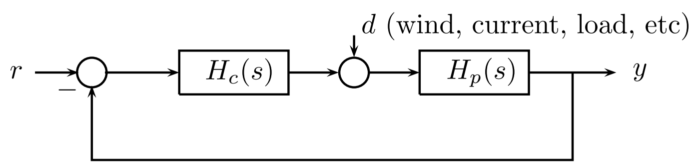


Solution: Augment integral control with lead control:

$$H_c(s) = \frac{K}{s}\frac{s-\beta}{s-\alpha} \quad \alpha < \beta < 0.$$

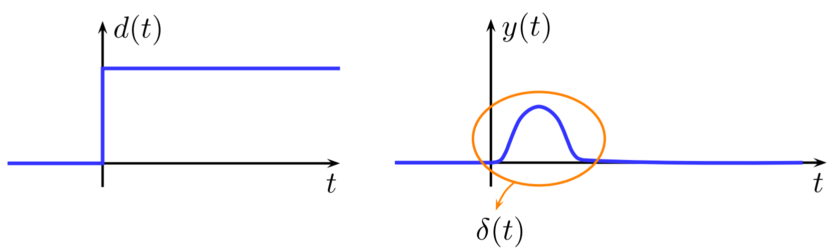
The main features of this controller are similar to PID (proportional-integral-derivative) control which is very popular in industry.

Disturbance Rejection with Integral Control



$$Y(s) = H_p(s)\left(H_c(s)(R(s) - Y(s)) + D(s)\right)$$
$$\left(1 + H_c(s)H_p(s)\right)Y(s) = H_c(s)H_p(s)R(s) + H_p(s)D(s)$$
$$Y(s) = \underbrace{\frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}R(s)}_{\triangleq Y_{\text{nominal}}(s)} + \underbrace{\frac{H_p(s)}{1 + H_c(s)H_p(s)}D(s)}_{\triangleq \Delta(s)}$$

Suppose $d(t) = u(t)$: unit step. How do we guarantee $y(t)$ recovers from this disturbance; that is, $\delta(t) \triangleq y(t) - y_{\text{nominal}}(t) \to 0$?



$$\Delta(s) = \frac{H_p(s)}{1 + H_c(s)H_p(s)}\frac{1}{s}$$

$$\lim_{t\to\infty}\delta(t) = \lim_{s\to 0} s\Delta(s) = \lim_{s\to 0}\frac{H_p(s)}{1 + H_c(s)H_p(s)}$$
$$\boxed{= 0 \text{ if } H_c(s) \text{ has a pole at } s = 0.}$$

Example: Consider again the position control example on page 1.
Although the plant $H_p(s) = \frac{1}{s(Ms+b)}$ has a pole at $s = 0$, a constant

gain controller $H_c(s) = K$ cannot eliminate the steady state offset caused by a disturbance entering the system before the plant (as in the block diagram above). Instead we get the offset:

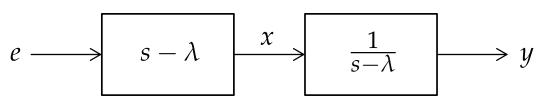
$$\lim_{t\to\infty} \delta(t) \quad = \quad \lim_{s\to 0} \frac{H_p(s)}{1 + H_c(s)H_p(s)} \quad = \quad \frac{1}{K}.$$

To remove the offset the control $H_c(s)$ itself must have a pole at $s = 0$.

What Happens to Canceled Poles?

They get decoupled from the input-output relation but continue to exist internally, creating dynamic modes that are invisible from the output or can't be influenced by the input.

As an illustration consider the series interconnection below where a pole-zero cancelation occurs at $s = \lambda$.



In the time domain the first and second blocks satisfy, respectively

$$x(t) = \frac{de(t)}{dt} - \lambda e(t) \quad \text{and} \quad \frac{dy(t)}{dt} = \lambda y(t) + x(t).$$

Combining the two, we get:

$$\frac{d}{dt}(y(t) - e(t)) = \lambda(y(t) - e(t)) \;\Rightarrow\; y(t) - e(t) = (y(0) - e(0))e^{\lambda t}.$$

Thus, instead of $y(t) = e(t)$, we have

$$y(t) = e(t) + (y(0) - e(0))e^{\lambda t}$$

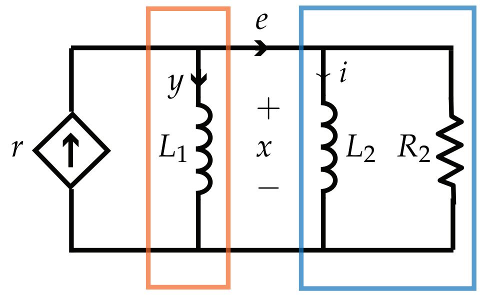
where the second term is a result of the canceled pole at $s = \lambda$. Since transfer functions don't account for initial conditions, this term does not appear in the transfer function of the series interconnection:

$$\frac{Y(s)}{E(s)} = (s - \lambda)\frac{1}{s - \lambda} = 1.$$

<u>Example:</u> Consider the circuit below where x is the voltage across the parallel interconnection of a current source with two inductors and a resistor. The currents r, e, y, and i are as labeled.

The orange block with input x and output y is governed by:

$$L_1 \frac{dy(t)}{dt} = x(t) \quad \Rightarrow \quad \frac{Y(s)}{X(s)} = \frac{1}{L_1 s}.$$

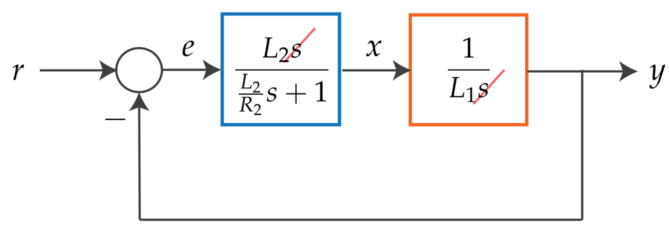


The blue block with input e and output x is governed by:

$$L_2 \frac{d}{dt} \underbrace{\left(e(t) - \frac{x(t)}{R_2} \right)}_{= i(t)} = x(t)$$

$$\Rightarrow \quad L_2 s \left(E(s) - \frac{X(s)}{R_2} \right) = X(s) \quad \Rightarrow \quad \frac{X(s)}{E(s)} = \frac{L_2 s}{\frac{L_2}{R_2} s + 1}.$$

Noting from Kirchoff's law that $e = r - y$, we view this circuit as a feedback interconnection of the two blocks with reference input r:



With the pole-zero cancelation at $s = 0$, the closed loop transfer function has a single pole at $s = -R_2(\frac{1}{L_1} + \frac{1}{L_2})$. Thus, when $r(t) \equiv 0$, one might expect the current $y(t)$ to decay to zero. However, this is not necessarily true: depending on the initial conditions, a constant current can remain in the loop formed by the two inductors as a result of the canceled pole at $s = 0$: