EE120 - Fall'15 - Lecture 21 Notes[1]

Murat Arcak

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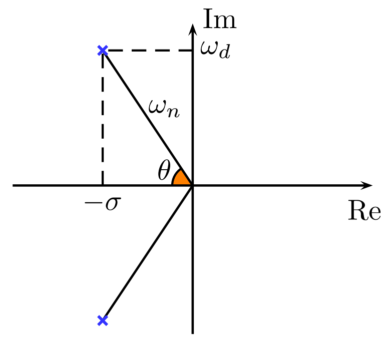
Step Response of Second Order Systems

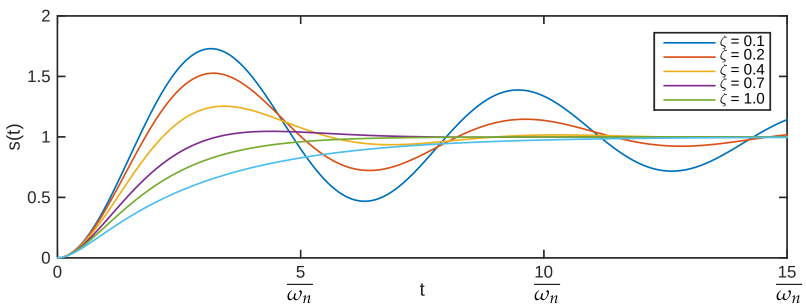
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ζ: damping ratio, ω_n: natural frequency

Poles: $s_{1,2} = \overbrace{-\omega_n cos\theta}^{\sigma} \mp \overbrace{j\omega_n sin\theta}^{\omega_d}$ where $\cos\theta = \zeta$



Below are the step responses for various values of ζ. Note that ω_n changes only the time scale, not the shape of the response.



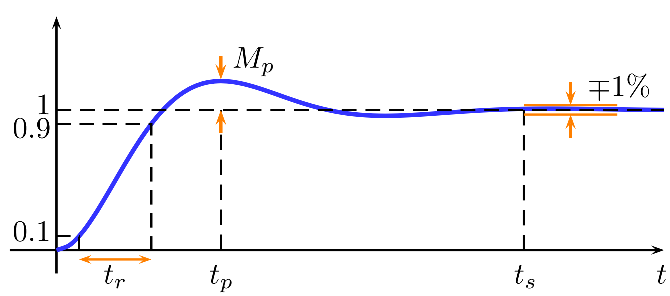
Important Features of the Step Response:

1) Rise time (tr): time to go from 10% to 90% of steady-state value

2) Peak overshoot (M_p): (peak value - steady state)/steady state

3) Peaking time (t_p): time to peak overshoot

4) Settling time (t_s): time after which the step response stays within 1% of the steady-state value



How do these parameters depend on ζ and ω_n?

$$u(t): \text{ unit step } \xleftrightarrow{\mathcal{L}} \frac{1}{s}$$

Step response:

$$\begin{aligned} Y(s) = \frac{1}{s}H(s) &= \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} \qquad (1) \\ &= \frac{A}{s} + \frac{B}{s+\sigma+j\omega_d} + \frac{B^*}{s+\sigma-j\omega_d} \\ & \quad A = 1 \quad B = -\frac{1}{2}\left(1+j\frac{\sigma}{\omega_d}\right) \end{aligned}$$

$$\begin{aligned} y(t) &= \left(1+Be^{-\sigma t}e^{-j\omega_d t}+B^*e^{-\sigma t}e^{j\omega_d t}\right)u(t) \\ &= \left(1+\underbrace{\left(Be^{-j\omega_d t}+B^*e^{j\omega_d t}\right)}_{\substack{=2Re\{\underbrace{Be^{-j\omega_d t}}_{=-\frac{1}{2}\left(1+j\frac{\sigma}{\omega_d}\right)(cos\omega_d t - jsin\omega_d t)}\} \\ = -\left(cos\omega_d t+\frac{\sigma}{\omega_d}sin\omega_d t\right)}}e^{-\sigma t}\right)u(t) \end{aligned}$$

$$\boxed{y(t) = \left[1-\left(cos\omega_d t+\frac{\sigma}{\omega_d}sin\omega_d t\right)e^{-\sigma t}\right]u(t)}$$

Peaking time:

$$\begin{aligned} \frac{d}{dt}y(t) &= \sigma e^{-\sigma t}\left(\cancel{cos\omega_d t}+\frac{\sigma}{\omega_d}sin\omega_d t\right) - e^{-\sigma t}\left(-\omega_d sin\omega_d t+\cancel{\sigma cos\omega_d t}\right) \\ &= e^{-\sigma t}\left(\frac{\sigma^2}{\omega_d}+\omega_d\right)sin\omega_d t \end{aligned}$$

$$\frac{d}{dt}y(t) = 0 \implies sin\omega_d t = 0 \qquad \boxed{t_p = \frac{\pi}{\omega_d}}$$

Peak overshoot: $M_p = y(t_p) - 1$

$$y(t_p) = \left(1-\underbrace{cos\omega_d t_p}_{=cos\pi=-1}e^{-\sigma t_p}\right) = 1+e^{-\sigma t_p} = 1+e^{-\sigma\frac{\pi}{\omega_d}}$$

$$\boxed{M_p = e^{-\pi\frac{\sigma}{\omega_d}} = e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}}} \qquad \begin{array}{l} \zeta \nearrow \implies M_p \searrow \\ M_p \to 0 \text{ as } \zeta \to 1 \end{array} \qquad M_p \approx \begin{cases} 0.05 & \zeta = 0.7 \\ 0.16 & \zeta = 0.5 \end{cases}$$

Approximate expressions for rise time and settling time:

$$\boxed{t_s \approx \frac{4.6}{\sigma}} \qquad \text{(obtained from } e^{-\sigma t_s} = 0.01)$$

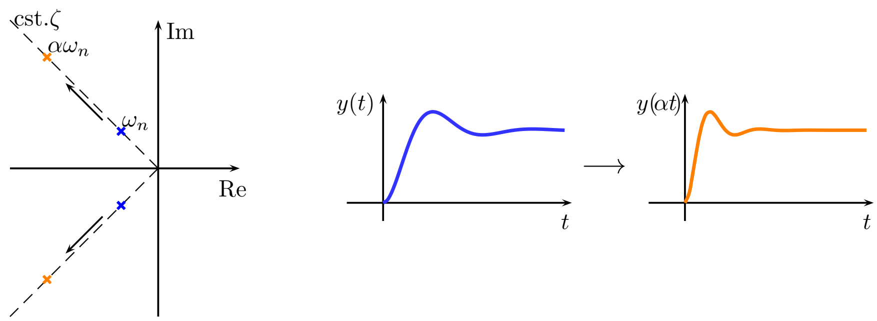
$$\boxed{t_r \approx \frac{1.8}{\omega_n}} \qquad \text{for } \zeta = 0.5 \text{ (changes little with } \zeta)$$

Note that t_p, t_s, t_r are inversely proportional to ω_n:

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \quad t_s \approx \frac{4.6}{\sigma} = \frac{4.6}{\omega_n \zeta} \quad t_r \approx \frac{1.8}{\omega_n}.$$

This is consistent with our observation on page 1 that ω_n changes only the time scale, not the shape of the response. We make this property explicit in the following statement:

If ζ is kept constant and ω_n is scaled by a factor of $\alpha > 0$ ($\omega_n \to \alpha\omega_n$) then the step response is scaled in time by α: $y(t) \to y(\alpha t)$.



Proof: If we replace ω_n with $\alpha\omega_n$ in (1), we get

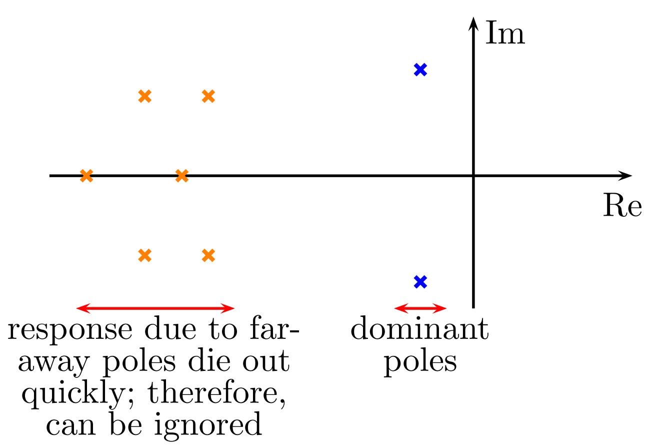
$$\frac{(\alpha\omega_n)^2}{s\left(s^2 + 2\zeta(\alpha\omega_n)s + (\alpha\omega_n)^2\right)} = \frac{\omega_n^2}{s\left(\left(\frac{s}{\alpha}\right)^2 + 2\zeta\omega_n\left(\frac{s}{\alpha}\right) + \omega_n^2\right)} = \frac{1}{\alpha}Y\left(\frac{s}{\alpha}\right).$$

The statement above then follows from the scaling property of Laplace transform:

$$y(\alpha t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{\alpha}Y\left(\frac{s}{\alpha}\right).$$

Summary: $\omega_n \nearrow$ increases speed of the response
$\zeta \nearrow$ reduces overshoot

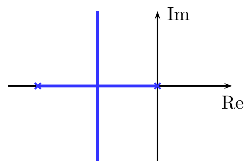
Although the formulas above are for second order systems, they can be applied as approximate expressions to higher order systems with two dominant poles:



Control Design by Root Locus

Root locus examples from last lecture:

1)



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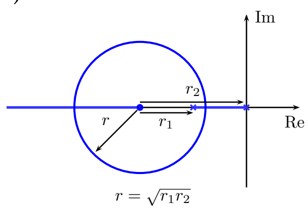
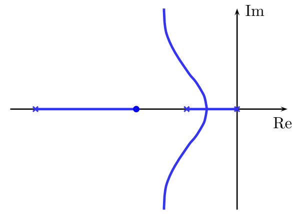
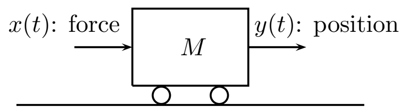


3)

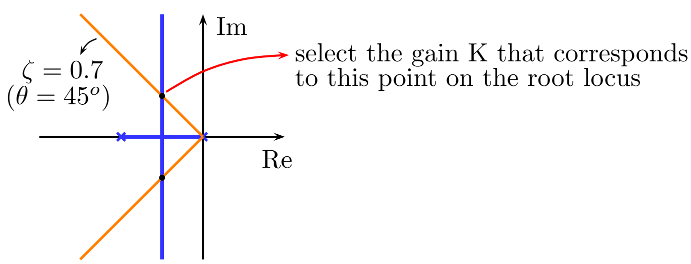


<u>Example:</u>

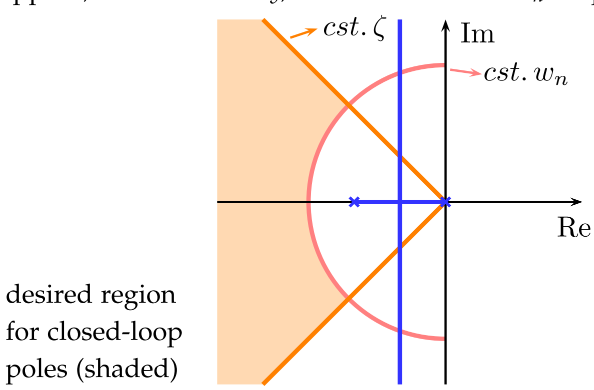


$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} = x(t) \quad \rightarrow \quad H_p(s) = \frac{1}{Ms^2 + bs}$$

Suppose a damping ratio of $\zeta = 0.7$ is desired:



Suppose, in addition to ζ, a lower bound on ω_n is specified:



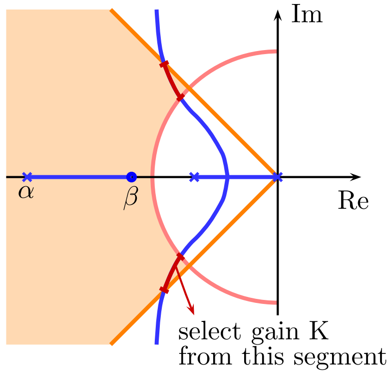
The root locus doesn't go through the desired region, therefore constant gain control won't work. Try the controller:

$$H_c(s) = K\frac{s-\beta}{s-\alpha} \quad \alpha < \beta < 0 \quad \text{(pole to the left of zero)}$$

Closed-loop poles:

$$1 + \underbrace{K\overbrace{\frac{s-\beta}{s-\alpha}}^{H_c(s)}\overbrace{\frac{1}{s(Ms+b)}}^{H_p(s)}}_{H(s)} = 0$$

Select α, β such that the root locus passes through the desired region



A controller of the form

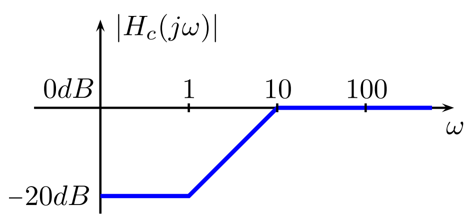
$$\boxed{H_c(s) = K\frac{s-\beta}{s-\alpha} \quad \alpha < \beta < 0}$$

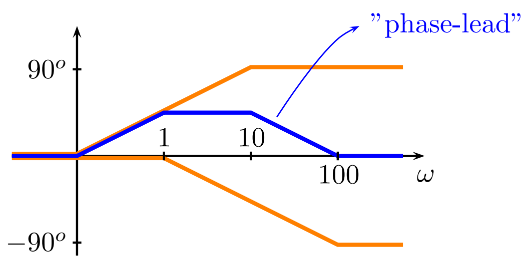
is called a "lead controller".

Example:

$$\begin{aligned} H_c(s) &= \frac{s+1}{s+10} \\ H_c(j\omega) &= \frac{1}{10}\frac{1+j\omega}{1+j\omega/10} \end{aligned}$$

$$20log_{10}|H_c(j\omega)| = -20 - 20log_{10}|1+j\omega/10| + 20log_{10}|1+j\omega|$$





Some History: Black's Feedback Amplifier

The use of feedback is not limited to designing controllers that shape the dynamic response of a system. Another important advantage is to guarantee robustness to variations and disturbances.

In its early days Bell Labs developed amplifiers that enabled long distance telephone communication. However, the amplifiers had significant variations in their gains and their nonlinearity caused interference between the channels. Addressing these problems Harold Black introduced a negative feedback around the amplifier that both reduced the variations in the gains and extended the linear range.

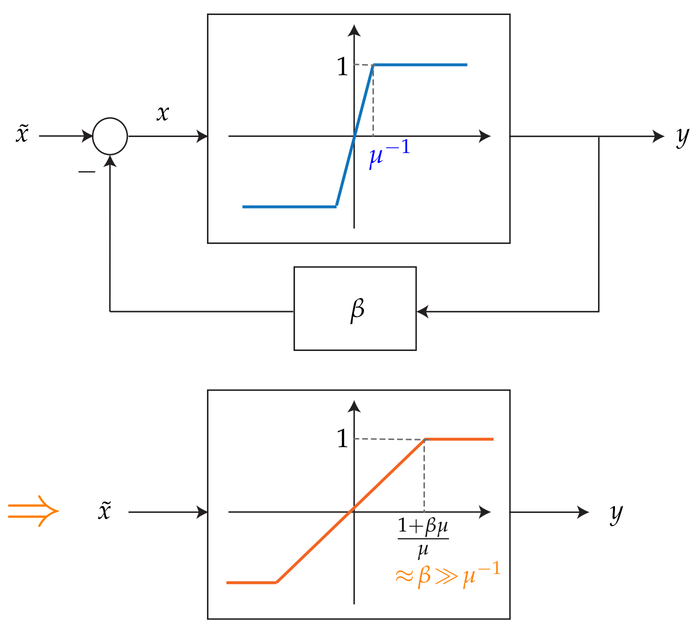
We illustrate these benefits on a static model of the amplifier in the figure below. Suppose the amplifier has gain μ in its linear range and the output saturates at ± 1. When a negative feedback with gain

$$\beta \gg \mu^{-1}$$

is applied, the relationship between the new input $\tilde{x}$ and the output y is again a saturation nonlinearity (show this), but the new gain is

$$\frac{\mu}{1+\beta\mu} \approx \beta^{-1}$$

which is robust to variations in μ. In addition, the response is linear when $|\tilde{x}| \leq \frac{1+\beta\mu}{\mu} \approx \beta$, a significantly wider range than $|x| \leq \mu^{-1}$.



The drawback is that the gain of the amplifier is now significantly reduced. As Black explains in his 1934 paper in the Bell System Technical Journal:

"... by building an amplifier whose gain is deliberately made say 40 decibels higher than necessary and then feeding the output back on the input in such a way as to throw away excess gain, it has been found possible to effect extraordinarily improvement in constancy of amplification and freedom from nonlinearity."