EE120 - Fall'15 - Lecture 2 Notes[1]

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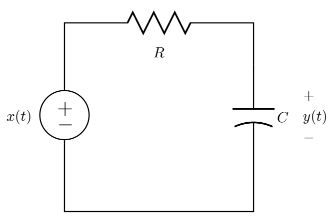
LTI Systems Described by Differential and Difference Equations

Linear Constant Coefficient Differential Equations:

$$x(t) \rightarrow \boxed{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}} \rightarrow y(t) \tag{1}$$

If the initial conditions are zero: $y(0) = \frac{dy}{dt}(0) = ... = \frac{dy^{N-1}}{dt^{N-1}}(0) = 0$, then this is an LTI system. Nonzero initial conditions destroy linearity: if $y(0) \neq 0$, then $x(t) \equiv 0$ does not imply $y(t) \equiv 0$.

Example:



$$C\frac{dy}{dt} = \frac{-y + x}{R} \tag{2}$$

With $x(t) \equiv 0$, solution is $y(t) = y(0)e^{-t/RC}$ ($\equiv 0$ only if $y(0) = 0$).

Linear Constant Coefficient Difference Equations:

$$x[n] \rightarrow \boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]} \rightarrow y[n] \tag{3}$$

LTI if $y[-1] = y[-2] = ... = y[-N] = 0$.

Example: Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k] \;\rightarrow\; y[n] - y[n-1] = x[n] \tag{4}$$

With $x[n] \equiv 0$, the solution is $y[n] = y[-1]$ for $n \geq 0$. □

Rewrite the difference equation (3) as:

$$\boxed{a_0 y[n] + a_1 y[n-1] + ... + a_N y[n-N] = b_0 x[n] + ... b_M x[n-M]} \tag{5}$$

and note that it defines a causal system if $a_0 \neq 0$.

Henceforth assume $a_0 = 1$ (if $a_0 \neq 1$, we can divide the other coefficients by a_0).

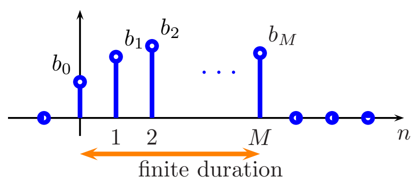
FIR vs. IIR Systems

The special case $N = 0$ above defines a *finite impulse response* (FIR) system:

$$y[n] = b_0 x[n] + ... + b_M x[n-M] \tag{6}$$

Impulse response:

$$h[n] = b_0\delta[n] + b_1\delta[n-1] + ... + b_M\delta[n-M] \tag{7}$$



Note that a FIR system is always stable, because the sum $\sum_n |h[n]|$ is over a finite duration and, thus, finite.

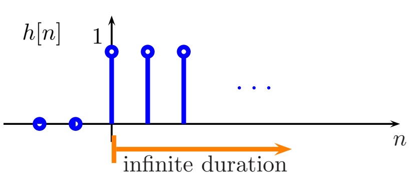
An *infinite impulse response* (IIR) example:

$y[n] - y[n-1] = x[n]$, $y[-1] = 0$ (accumulator)

Impulse response:

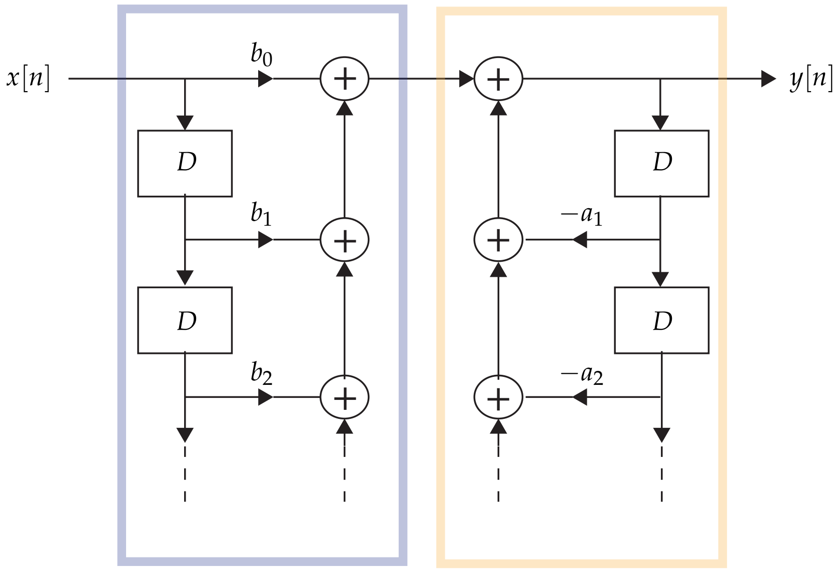
$$h[n] - h[n-1] = \delta[n]$$

$$\begin{aligned} h[0] = h[-1] + \delta[0] &= 0 + 1 = 1 \\ h[1] = h[0] &= 1 \\ h[2] = h[1] &= 1 \\ &\vdots \end{aligned}$$

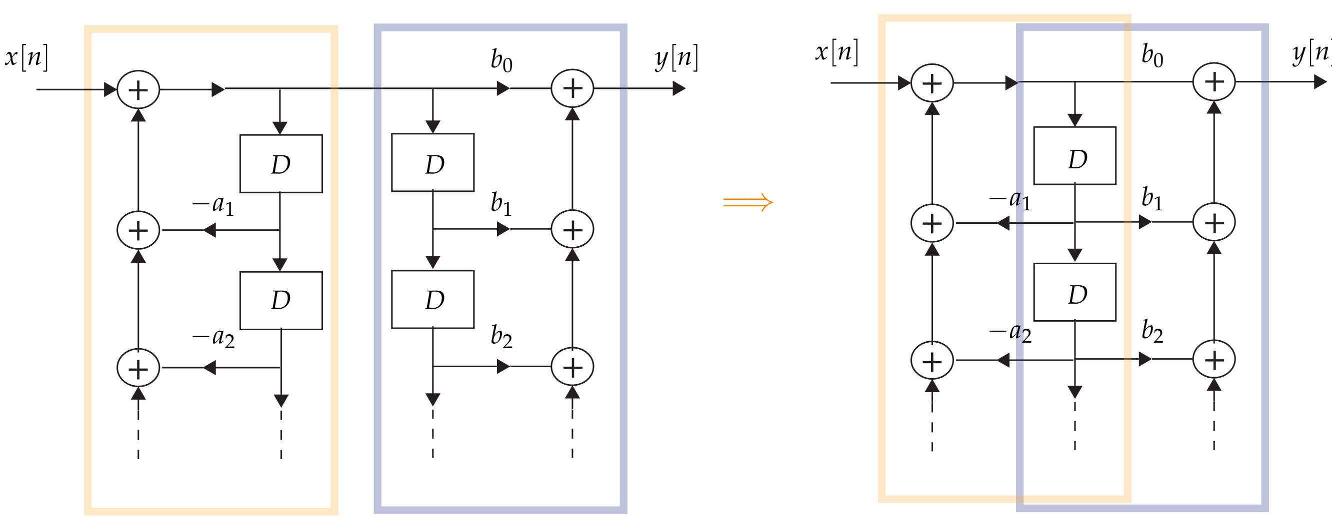


Block Diagram Representation of DT LTI Systems:

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \tag{8}$$



This requires $N + M$ delay elements (memory registers). For an implementation with fewer memory registers, recall that changing the order of two LTI systems in series does not change the result:



Note that a FIR system can be implemented with the blue block only; no feedback loops are required.

Response of LTI Systems to Complex Exponentials

Section 3.2 *in* Oppenheim & Willsky

Complex Exponentials

Continuous-time:

$$x(t) = e^{st},\ s \in \mathbb{C} \xrightarrow{s=\sigma+j\omega} x(t) = \underbrace{e^{\sigma t}}_{\text{envelope}} \underbrace{e^{j\omega t}}_{\text{periodic}} \tag{9}$$

Discrete-time:

$$x[n] = z^n,\ z \in \mathbb{C} \xrightarrow{z=re^{j\omega}} x[n] = r^n e^{j\omega n} \tag{10}$$

Figures 1 and 2 on page 7 plot e^{st} and z^n for various values of s and z in the complex plane.

> The response of a LTI system to a complex exponential is the same complex exponential scaled by a constant.

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = \underbrace{\left(\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\right)}_{\triangleq H(s)} e^{st} \tag{11}$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = \underbrace{\left(\sum_{-\infty}^{\infty} h[k]z^{-k}\right)}_{\triangleq H(z)} z^n \tag{12}$$

$H(s)$ and $H(z)$ are called "transfer functions" or "system functions."

<u>Example:</u> Find the transfer function $H(s)$ for $y(t) = x(t-3)$.
<u>If $x(t) = e^{st}$</u> then

$$y(t) = x(t-3) = e^{s(t-3)} = \underbrace{e^{-3s}}_{=H(s)} e^{st}. \tag{13}$$

Alternatively, use the impulse response $h(t) = \delta(t-3)$:

$$H(s) = \int_{-\infty}^{\infty} \delta(\tau-3)e^{-s\tau}d\tau = e^{-3s}. \tag{14}$$

Frequency Response of a LTI System

$$\begin{aligned} x(t) = e^{j\omega t}(s = j\omega) \quad &\rightarrow \quad y(t) = H(j\omega)e^{j\omega t} \\ x[n] = e^{j\omega n}(z = e^{j\omega}) \quad &\rightarrow \quad y[n] = H(e^{j\omega})e^{j\omega n} \end{aligned} \tag{15}$$

$$\boxed{\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \\ H(e^{j\omega}) &= \sum h[k]e^{-j\omega k} \end{aligned}} \tag{16}$$

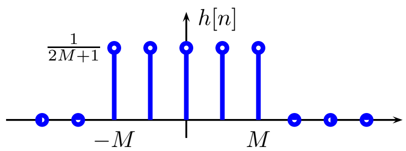
Filtering

Section 3.9 *in* Oppenheim & Willsky

LTI system designed such that $H(j\omega)$ ($H(e^{j\omega})$ in DT) is zero or close to zero for frequencies to be eliminated.

Example: Why is the moving average system a low-pass filter?

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k] \tag{17}$$

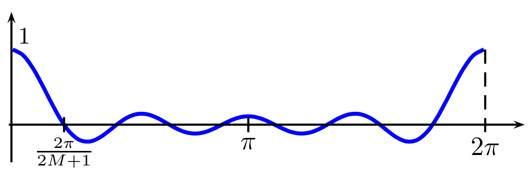


$$H(e^{j\omega}) = \sum_{k=-M}^{M} \frac{1}{2M+1} e^{-j\omega k} = \frac{e^{j\omega M}}{2M+1} \underbrace{\left(1 + e^{-j\omega} + ... + e^{-j\omega 2M}\right)}_{\frac{1-e^{-j\omega(2M+1)}}{1-e^{-j\omega}} \text{ if } w \neq 0^{2}}$$

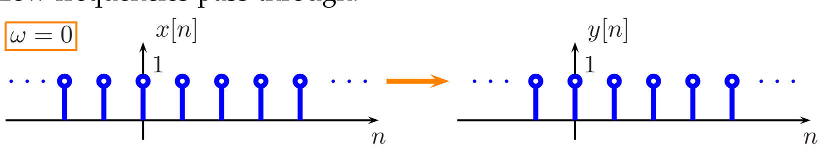
$$= \frac{1}{2M+1} e^{j\omega M} \underbrace{\frac{e^{-j\omega(M+\frac{1}{2})}}{e^{-j\omega/2}}}_{=1} \underbrace{\frac{e^{j\omega(M+\frac{1}{2})} - e^{-j\omega(M+\frac{1}{2})}}{e^{j\omega/2} - e^{-j\omega/2}}}_{\frac{\sin(\omega(M+1/2))}{\sin(\omega/2)}}$$

[2] since this is a geometric series

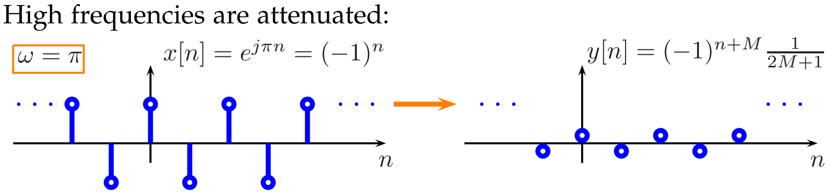
$$H(e^{j\omega}) = \begin{cases} 1 & \text{if } \omega = 0 \\ \frac{1}{2M+1} \frac{\sin(\omega(M+1/2))}{\sin(\omega/2)} & \omega \neq 0 \end{cases} \tag{18}$$



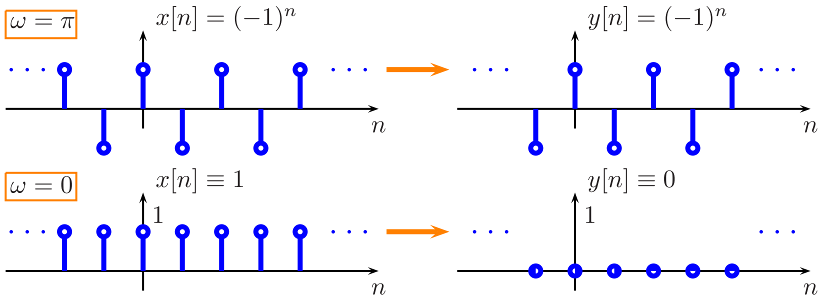
Low frequencies pass through:



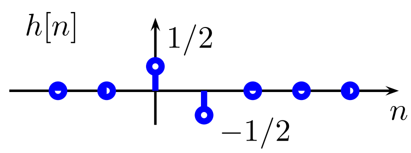
High frequencies are attenuated:



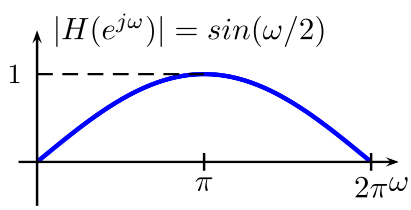
Example: Is $y[n] = \frac{1}{2}(x[n] - x[n-1])$ low-pass or high-pass?



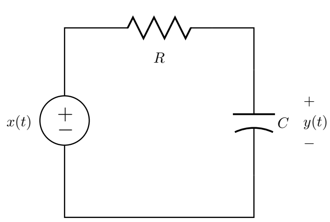
To find $H(e^{j\omega})$, note that the impulse response is:



$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \frac{1}{2} - \frac{1}{2}e^{-j\omega} = \frac{1}{2}(1 - e^{-j\omega}) = \frac{1}{2}e^{-j\omega/2}2jsin(\omega/2) \tag{19}$$



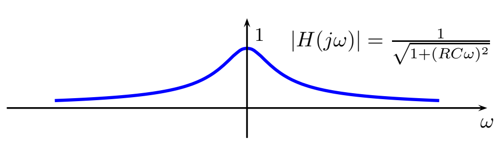
Example: CT low-pass filter



$$RC\frac{dy}{dt} + y = x \quad y(0) = 0$$
$$x = e^{j\omega t} \rightarrow y = H(j\omega)e^{j\omega t}$$
$$RC\frac{d}{dt}\left\{H(j\omega)e^{j\omega t}\right\} + H(j\omega)e^{j\omega t} = e^{j\omega t}$$
$$j\omega RCH(j\omega) + H(j\omega) = 1$$

Therefore,

$$H(j\omega) = \frac{1}{1 + j\omega RC} \tag{20}$$



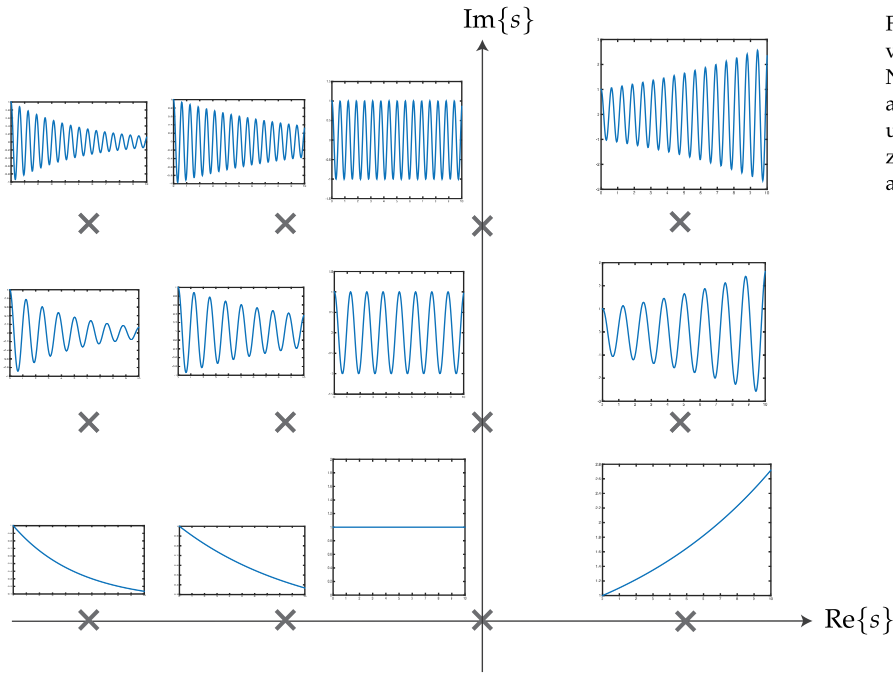


Figure 1: The real part of e^{st} for various values of s in the complex plane. Note that e^{st} is oscillatory when s has an imaginary component. It grows unbounded when $\text{Re}\{s\} > 0$, decays to zero when $\text{Re}\{s\} < 0$, and has constant amplitude when $\text{Re}\{s\} = 0$.

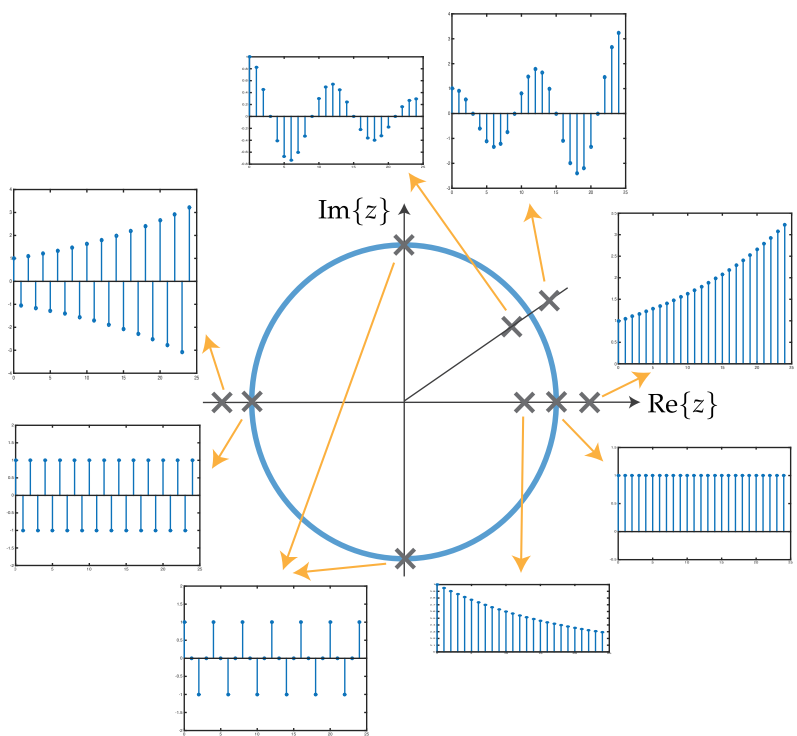


Figure 2: The real part of z^n for various values of z in the complex plane. It grows unbounded when $|z| > 1$, decays to zero when $|z| < 1$, and has constant amplitude when z is on the unit circle ($|z| = 1$).