

# EE120 - Fall'15 - Lecture 19 Notes<sup>1</sup>

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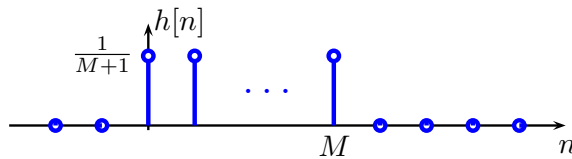
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## Geometric Evaluation of the Frequency Response (Continued)

Example:  $(M + 1)$ -point moving average system

$$y[n] = \frac{1}{M+1} (x[n] + x[n-1] + \dots + x[n-M])$$



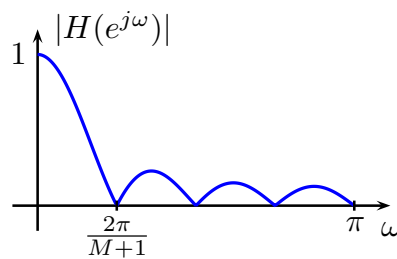
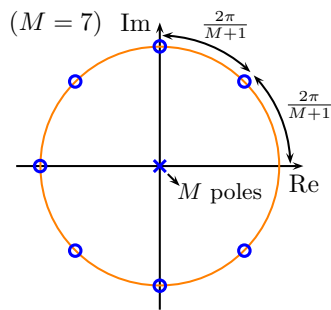
$$\begin{aligned} H(z) &= \frac{1}{M+1} (1 + z^{-1} + \dots + z^{-M}) \\ &= \frac{1}{M+1} \frac{z^M + z^{M-1} + \dots + 1}{z^M} \end{aligned}$$

All poles at  $z = 0$ . (Note this is true for any FIR system.)

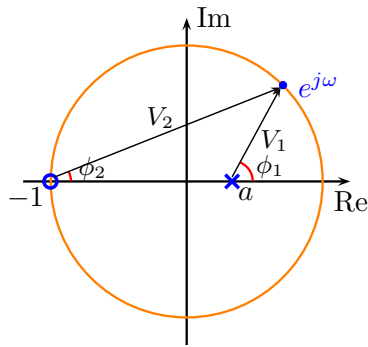
Zeros: roots of  $z^M + z^{M-1} + \dots + 1$ .

From the identity  $z^{M+1} - 1 = (z - 1)(z^M + z^{M-1} + \dots + 1)$ , the roots of  $z^M + z^{M-1} + \dots + 1$  are the roots of  $z^{M+1} - 1$  except for  $z = 1$ .

$$z^{M+1} = 1 \implies z = e^{j\frac{2\pi}{M+1}k} \quad k = 1, 2, \dots, M \quad (k = 0, \text{ i.e., } z = 1 \text{ excluded})$$



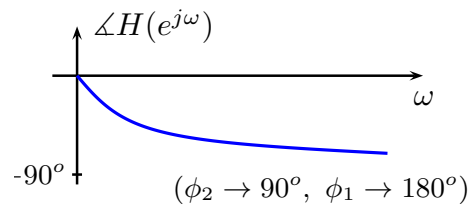
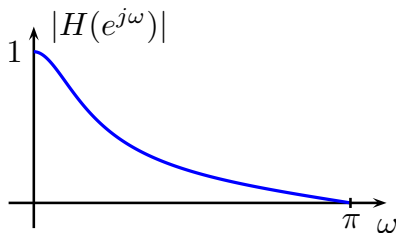
Example: Finding the phase



$$H(z) = \frac{1-a}{2} \frac{1+z^{-1}}{1-az^{-1}} = \frac{1-a}{2} \frac{z+1}{z-a}$$

$$H(e^{j\omega}) = \frac{1-a}{2} \frac{e^{j\omega}+1}{e^{j\omega}-a} = \frac{1-a}{2} \frac{V_2}{V_1}$$

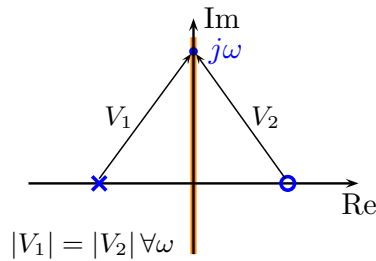
$$|H(e^{j\omega})| = \frac{1-a}{2} \frac{|V_2|}{|V_1|} \quad \angle H(e^{j\omega}) = \phi_2 - \phi_1$$



All Pass Systems

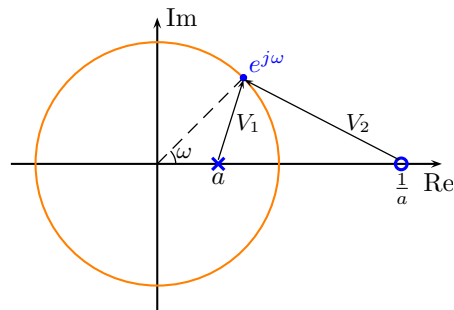
Continuous-time:

$$H(s) = \frac{a-s}{s+a}$$



Discrete-time:

$$H(z) = \frac{z^{-1}-a}{1-az^{-1}} = -a \frac{z-1/a}{z-a}, \quad a \text{ real}$$



$$|V_1|^2 = (\cos\omega - a)^2 + \sin^2\omega = 1 - 2a\cos\omega + a^2$$

$$|V_2|^2 = \left(\frac{1}{a} - \cos\omega\right)^2 + \sin^2\omega = \frac{1}{a^2} - \frac{2}{a}\cos\omega + 1$$

$$\text{then, } a^2|V_2|^2 = |V_1|^2 \Rightarrow |a| \frac{|V_2|}{|V_1|} = 1.$$

$$|H(e^{j\omega})| = |a| \frac{|e^{j\omega} - 1/a|}{|e^{j\omega} - a|} = |a| \frac{|V_2|}{|V_1|} = 1 \quad \forall \omega$$

General form of a DT all pass system:

$$H(z) = \prod_{k=1}^N \left( \frac{z^{-1} - a_k}{1 - a_k z^{-1}} \right)$$

Each pole  $a_k$  accompanied by a zero at  $1/a_k$ .

For a stable and causal all pass system, all zeros are outside the unit circle.

### Minimum Phase Systems

**Definition:** A stable and causal DT LTI system with transfer function  $H(z)$  whose zeros are also within the unit circle.

As in continuous-time, a nonminimum phase transfer function  $H(z)$  can be decomposed as:

$$H(z) = H_{min}(z)H_{ap}(z)$$

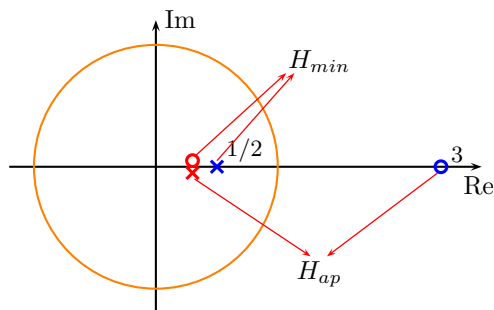
where  $H_{min}(z)$  is minimum phase and  $H_{ap}(z)$  is all pass.

Construct  $H_{ap}(z)$  such that it encompasses all zeros of  $H(z)$  outside the unit circle. Obtain  $H_{min}(z)$  from  $H(z)/H_{ap}(z)$ .

Example:

$$H(z) = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}} \quad \text{zero @ 3 outside the unit circle}$$

$$H_{ap}(z) = \frac{z^{-1} - 1/3}{1 - \frac{1}{3}z^{-1}} \quad H_{min}(z) = \frac{H(z)}{H_{ap}(z)} = \frac{1 - 3z^{-1}}{1 - \frac{1}{2}z^{-1}} \frac{1 - \frac{1}{3}z^{-1}}{z^{-1} - 1/3} = -3 \frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

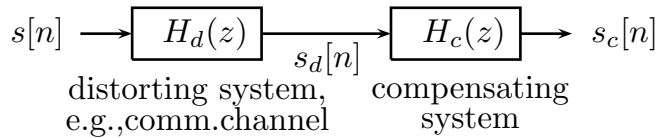


Pole/zero pair added at  $1/3$ .

Pole at  $1/3$  paired with existing zero at  $3$  to form all pass system.

Zero at  $1/3$  paired with existing pole at  $1/2$  to form minimum phase system.

Example:



If  $H_d(z)$  is minimum phase, we can design  $H_c(z) = \frac{1}{H_d(z)}$  (stable).

If not, decompose:  $H_d(z) = H_{d,min}(z)H_{d,ap}(z)$

and design:  $H_c(z) = \frac{1}{H_{d,min}(z)} \implies H_d(z)H_c(z) = H_{d,ap}(z)$

Magnitude distortion corrected.

### The Unilateral z-Transform

Section 10.9 in Oppenheim & Willsky

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots \quad (1)$$

### Properties of the Unilateral z-Transform

Most properties of the bilateral z-transform hold for the unilateral transform.

Exceptions:

Convolution:

$$x_1[n] * x_2[n] \xleftrightarrow{\mathcal{U}\mathcal{Z}} \mathcal{X}_1(z)\mathcal{X}_2(z) \quad \text{if } x_1[n] = x_2[n] = 0 \forall n < 0.$$

Time Delay:

$$x[n-1] \xleftrightarrow{\mathcal{U}\mathcal{Z}} z^{-1}\mathcal{X}(z) + x[-1]$$

Contrast to:  $x[n-1] \xleftrightarrow{\mathcal{Z}} z^{-1}\mathcal{X}(z)$

Proof:

$$\begin{aligned} \sum_{n=0}^{\infty} x[n-1]z^{-n} &= x[-1] + \underbrace{x[0]z^{-1} + x[1]z^{-2} + x[2]z^{-3} + \dots}_{=z^{-1}(x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots)} \\ &= z^{-1}\mathcal{X}(z) + x[-1] \end{aligned}$$

Applying repeatedly:

$$\begin{aligned} x[n-2] &\xleftrightarrow{\mathcal{U}\mathcal{Z}} z^{-1}(z^{-1}\mathcal{X}(z) + x[-1]) + x[-2] \\ &= z^{-2}\mathcal{X}(z) + x[-1]z^{-1} + x[-2] \\ x[n-3] &\xleftrightarrow{\mathcal{U}\mathcal{Z}} z^{-1}(z^{-2}\mathcal{X}(z) + x[-1]z^{-1} + x[-2]) + x[-3] \\ &= z^{-3}\mathcal{X}(z) + x[-1]z^{-2} + x[-2]z^{-1} + x[-3] \end{aligned}$$

*Solving Difference Equations using the Unilateral z-Transform*

Example:  $y[n] - 0.6y[n-1] = (0.5)^n u[n]$

Take unilateral z-transforms on both sides:

$$Y(z) - 0.6(z^{-1}Y(z) + y[-1]) = \frac{1}{1 - 0.5z^{-1}}$$

$$(1 - 0.6z^{-1})Y(z) = 0.6y[-1] + \frac{1}{1 - 0.5z^{-1}} = \frac{1 + 0.6y[-1] - 0.3y[-1]z^{-1}}{1 - 0.5z^{-1}}$$

$$Y(z) = \frac{(1 + 0.6y[-1]) - 0.3y[-1]z^{-1}}{(1 - 0.6z^{-1})(1 - 0.5z^{-1})} = \frac{A}{1 - 0.6z^{-1}} + \frac{B}{1 - 0.5z^{-1}}$$

$$\left. \begin{aligned} A + B &= 1 + 0.6y[-1] \\ 0.5A + 0.6B &= 0.3y[-1] \end{aligned} \right\} \begin{aligned} B &= -5 \\ A &= 6 + 0.6y[-1] \end{aligned}$$

$$y[n] = (6 + 0.6y[-1])(0.6)^n u[n] - 5(0.5)^n u[n]$$

Compare to the time domain method:

1) Homogenous solution:  $A(0.6)^n$

2) Particular solution:  $y_p[n] = B(0.5)^n$

Substitute in difference equation to find B:

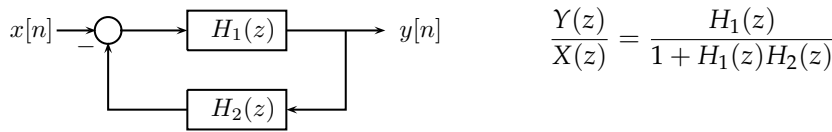
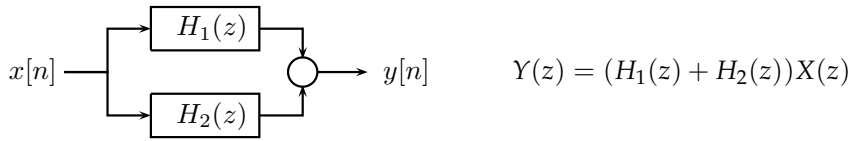
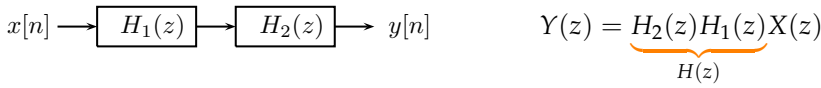
$$\begin{aligned} B(0.5)^n - 0.6B(0.5)^{n-1} &= (0.5)^n \\ B - 0.6(0.5)^{-1}B &= 1 \implies -0.2B = 1 \implies B = -5 \end{aligned}$$

3) The complete solution is  $y[n] = A(0.6)^n - 5(0.5)^n$  and

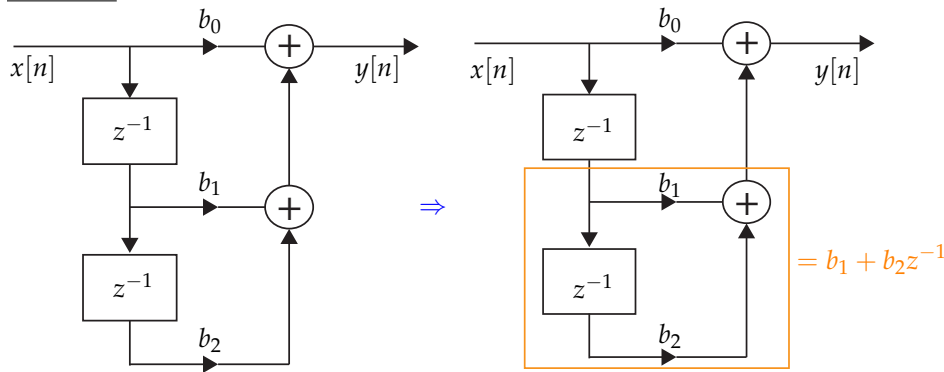
A is determined from the initial condition:

$$\begin{aligned} y[-1] &= A(0.6)^{-1} - 5(0.5)^{-1} \\ A &= (0.6)(y[-1] + 10) = 6 + 0.6y[-1] \end{aligned}$$

Interconnections of DT LTI Systems

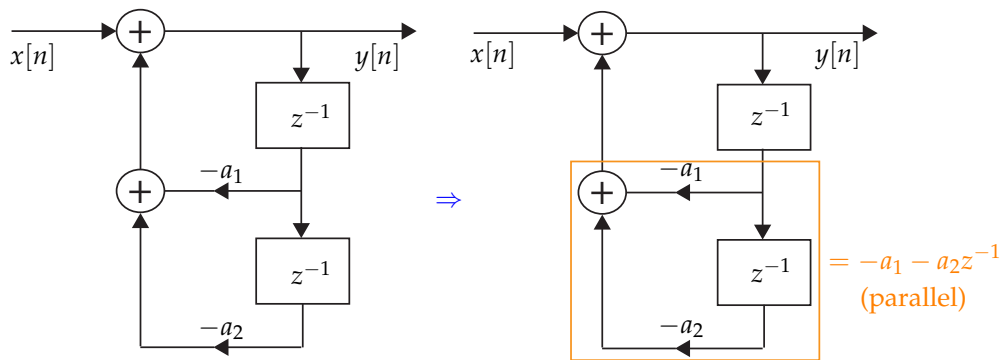


Example:



$\Rightarrow H(z) = b_0 + z^{-1}(b_1 + b_2z^{-1}) = b_0 + b_1z^{-1} + b_2z^{-2}$

Example:



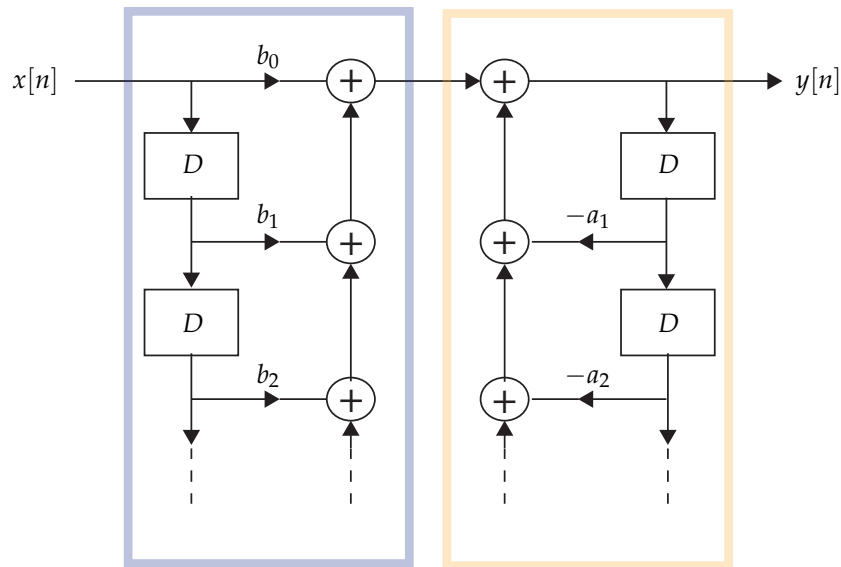
Then, from the feedback interconnection formula:

$$H(z) = \frac{1}{1 - z^{-1}(-a_1 - a_2z^{-1})} = \frac{1}{1 + a_1z^{-1} + a_2z^{-2}}$$

Recall the block diagram in Lecture 2 for:

$$y[n] = - \sum_{k=1}^N a_k y[n-k] + \sum_{k=0}^M b_k x[n-k]$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$



The delay element  $D$  corresponds to  $z^{-1}$ . The blue block implements the numerator:

$$H_1(z) = b_0 + b_1 z^{-1} + \dots + b_M z^{-M}$$

and the orange block implements the denominator:

$$H_2(z) = \frac{1}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

as in the examples above. The series interconnection of the two gives:

$$H(z) = H_1(z)H_2(z).$$

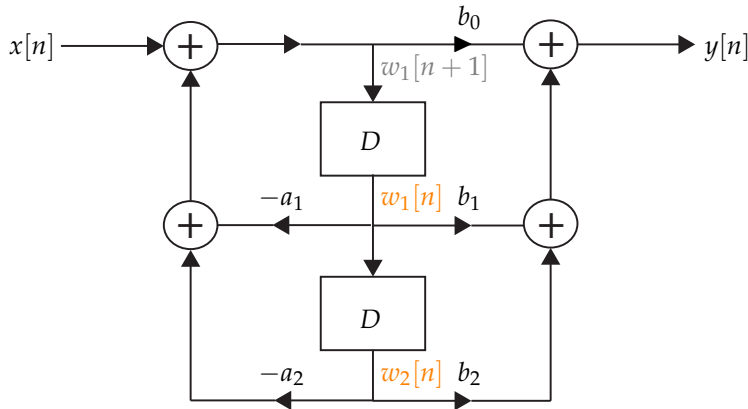
Changing the order of  $H_1(z)$  and  $H_2(z)$  does not change the product and allows us to use fewer delay elements, as was done in Lecture 2. (See figure on next page.)

### Obtaining Transfer Functions from Block Diagrams

In the examples above a repeated application of the series, parallel, and feedback interconnection formulas gave the transfer function. However, this is not an efficient approach in general and may not be possible for every interconnection.

We outline an alternative procedure and illustrate it on the block diagram below.

**Step 1:** Label the output of each delay element as a function of time:  $w_1[n], w_2[n], \dots$  These are called "state variables."



**Step 2:** Note that the input signal to the  $i$ th delay element is  $w_i[n+1]$  so that its output is  $w_i[n]$ . By inspecting the interconnection, express these inputs in terms of the input  $x[n]$  and state variables  $w_1[n], w_2[n], \dots$  In the block diagram above:

$$\begin{aligned} w_1[n+1] &= -a_1 w_1[n] - a_2 w_2[n] + x[n] \\ w_2[n+1] &= w_1[n] \end{aligned}$$

**Step 3:** Take the  $z$  transform of these equations and solve for  $W_1(z)$ ,  $W_2(z)$ , ... in terms of  $X(z)$ .

$$\begin{aligned} zW_1(z) &= -a_1 W_1(z) - a_2 W_2(z) + X(z) \\ zW_2(z) &= W_1(z) \end{aligned}$$

Substituting  $W_2(z) = W_1(z)/z$  in the first equation and rearranging:

$$\begin{aligned} \left(z + a_1 + \frac{a_2}{z}\right) W_1(z) = X(z) &\Rightarrow W_1(z) = \frac{z}{z^2 + a_1 z + a_2} X(z) \\ W_2(z) &= \frac{1}{z^2 + a_1 z + a_2} X(z) \end{aligned}$$

**Step 4:** Express  $Y(z)$  in terms of  $X(z)$  and  $W_1(z)$ ,  $W_2(z)$ , ..., again using the interconnection. Then substitute  $W_1(z)$ ,  $W_2(z)$ , ... from the previous step so  $Y(z)$  depends on  $X(z)$  only.

$$\begin{aligned} Y(z) &= b_0 z W_1(z) + b_1 W_1(z) + b_2 W_2(z) = \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} X(z) \\ H(z) &= \frac{b_0 z^2 + b_1 z + b_2}{z^2 + a_1 z + a_2} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}. \end{aligned}$$



Exercise: Show that the block diagram below yields the same transfer function

$$H(z) = \frac{b_0 + b_1z^{-1} + b_2z^{-2}}{1 + a_1z^{-1} + a_2z^{-2}}.$$

