# *EE120 - Fall'15 - Lecture 18 Notes*[1]

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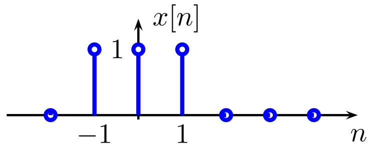
## *Properties of the z-Transform*

1) Linearity: $ax_1[n] + bx_2[n] \longleftrightarrow aX_1(z) + bX_2(z)$

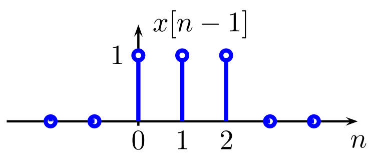
ROC contains $R_1 \cap R_2$ where $R_i$ is the ROC of $x_i[n]$, $i = 1, 2$.

2) Time Shifting: $x[n - n_0] \longleftrightarrow z^{-n_0} X(z)$

ROC unchanged, except for possible addition/deletion of 0 and $\infty$.




$\leftrightarrow$ $X(z) = z + 1 + z^{-1}$
ROC excludes 0 and $\infty$.




$\leftrightarrow$ $X(z) = 1 + z^{-1} + z^{-2} = z^{-1}X(z)$
ROC now includes $|z| = \infty$.

Example: Find the inverse z-Transform (right-sided) of

$$X(z) = \frac{1}{z^{-1}\left(1 - \frac{1}{2}z^{-1}\right)} = z\frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$x[n] = \left(\frac{1}{2}\right)^{n+1} u[n+1]$$

3) Scaling in the z-domain:

$$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{z}{z_0}\right) \quad \text{ROC} = |z_0| \cdot R$$

where $R$ is the ROC of $x[n]$. Compare to:

$$e^{j\omega_0 n} x[n] \overset{DTFT}{\longleftrightarrow} X\left(e^{j(\omega - \omega_0)}\right)$$

$$e^{s_0 t} x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0) \quad \text{ROC} = R + Re\{s_0\}$$

4) Time Reversal:

$$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X\left(\frac{1}{z}\right) \quad \text{ROC} = 1/R$$

Example:

$$x[n] = \left(\frac{1}{2}\right)^n u[n] \quad \leftrightarrow \quad X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2}$$

$$x[-n] = 2^n u[-n] \quad \leftrightarrow \quad X\left(\frac{1}{z}\right) = \frac{1}{1-\frac{1}{2}z} = \frac{-2z^{-1}}{1-2z^{-1}} \quad |z| < 2$$

5) Convolution Property:

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z)X_2(z) \quad \text{ROC contains } R_1 \cap R_2$$

6) Differentiation in z-domain:

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz} \quad \text{ROC unchanged}$$

Example:

$$na^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2} \quad |z| > |a|$$

7) Initial Value Theorem: If $x[n] = 0$ for $n < 0$, then

$$\boxed{x[0] = \lim_{z\to\infty} X(z)} \tag{1}$$

Proof: $X(z) = x[0] + \underbrace{x[1]z^{-1} + x[2]z^{-2} + ...}_{\to 0 \text{ as } z\to\infty}$

## *DT LTI Systems*

Section 10.7 *in* Oppenheim & Willsky

$$x[n] \to \boxed{h[n]} \to y[n] = h[n] * x[n]$$

$$Y(z) = \underbrace{H(z)}_{\text{transfer function}}X(z) \quad \to \quad H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Causality: $h[n] = 0 \ \forall n < 0$. Stability: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

### *Determining causality from $H(z)$*

Recall: For a right-sided $x[n]$ with rational $X(z)$, ROC extends from the outermost pole to infinity (infinity included if $x[n] = 0 \ \forall n < 0$).

A DT LTI system with rational transfer function $H(z)$ is causal if and only if the ROC extends from the outermost pole and includes $|z| = \infty$.

Examples:
$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}$$

ROC can't include $|z| = \infty$ because the numerator has higher order than the denominator: $3 > 2 \implies$ not causal

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} \qquad \text{ROC: } |z| > 2 \implies \text{causal}$$

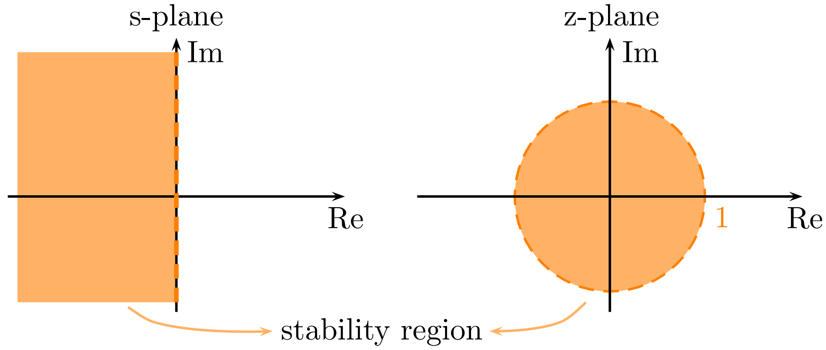
*Stability*

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \quad \text{means that the ROC includes the unit circle.}$$

An LTI system is stable if and only if the ROC of transfer function $H(z)$ includes the unit circle.

Sharper condition if the system is causal and $H(z)$ rational:

A causal LTI system with rational transfer function $H(z)$ is stable if and only if all poles of $H(z)$ lie inside the unit circle.




*From Difference Equations to Transfer Functions:*

$$\begin{aligned}
\sum_{k=0}^{N} a_k y[n-k] &= \sum_{k=0}^{M} b_k x[n-k] \\
\sum_{k=0}^{N} a_k z^{-k} Y(z) &= \sum_{k=0}^{M} b_k z^{-k} X(z) \\
H(z) = \frac{Y(z)}{X(z)} &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}
\end{aligned}$$

Example:

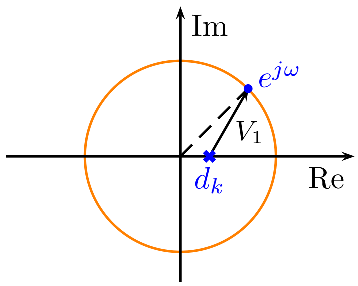
$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1] \implies H(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

*Geometric Evaluation of the Frequency Response $H(e^{j\omega})$:*

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})} \qquad \begin{matrix} c_k : \text{zeros} \\ d_k : \text{poles} \end{matrix}$$

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}|1 - c_k e^{-j\omega}|}{\prod_{k=1}^{N}|1 - d_k e^{-j\omega}|}$$

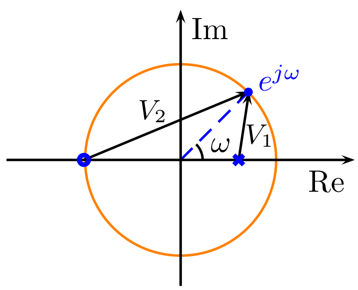
$$|1 - d_k e^{-j\omega}| = |e^{j\omega} - d_k| = |V_1|$$

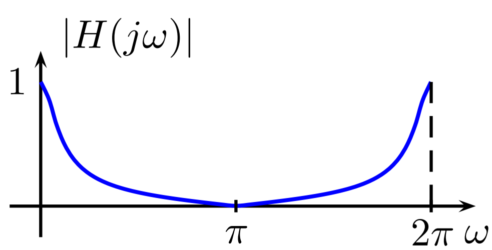



Example:

$$H(z) = \underbrace{0.05}_{\text{s.t. } H(1)=1 \text{ (dc gain)}} \frac{1 + z^{-1}}{1 - 0.9z^{-1}}$$

$$|H(e^{j\omega})| = 0.05 \frac{|V_2|}{|V_1|}$$

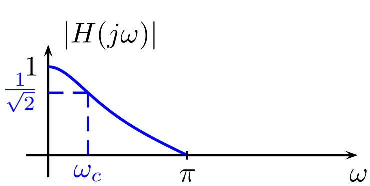



*Simple Filters:*

Low-Pass:

$$H(z) = \frac{1-\alpha}{2} \frac{1 + z^{-1}}{1 - \alpha z^{-1}}, \quad |\alpha| < 1 \text{ for stability}$$

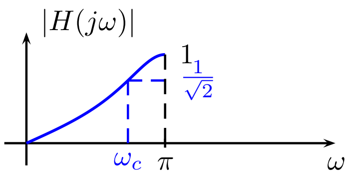



3dB cutoff frequency $\omega_c$ related to $\alpha$ by:

$$\alpha = \frac{1 - sin(\omega_c)}{cos(\omega_c)}$$

High-Pass:

$$H(z) = \frac{1+\alpha}{2}\frac{1-z^{-1}}{1-\alpha z^{-1}}, \quad H(1) = 0 \text{ and } H(-1) = H(e^{j\pi}) = 1$$

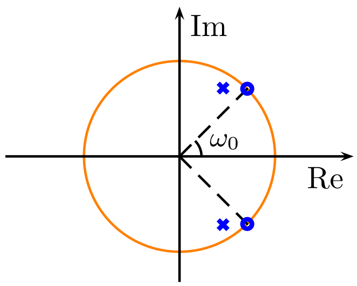



$$\alpha = \frac{1 - sin(\omega_c)}{cos(\omega_c)}$$

Band-Stop (Notch):

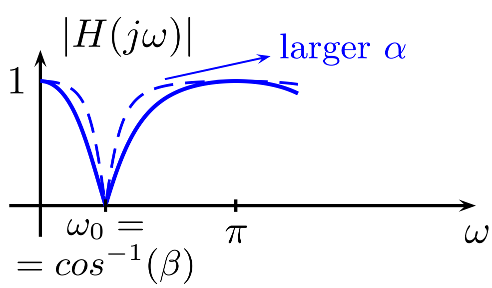
$$H(z) = \frac{1+\alpha}{2}\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}} \quad |\beta| < 1 \;\; |\alpha| < 1$$

Note: $1 - 2\beta z^{-1} + z^{-2} = (1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1})$ where $cos\omega_0 = \beta$




zeros on the unit circle: $e^{\mp j\omega_0}$
poles approach zeros as $\alpha \to 1$.

$$H(\mp 1) = \frac{1+\alpha}{2}\frac{2 \pm 2\beta}{(1+\alpha)(1 \pm \beta)} = 1$$




Band-Pass:

$$H(z) = \frac{1-\alpha}{2}\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}, \quad |\alpha| < 1 \;\; |\beta| < 1$$