EE120 - Fall'15 - Lecture 17 Notes[1]

Murat Arcak

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The z-Transform

Chapter 10 *in* Oppenheim & Willsky

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{1}$$

$$X(z)\Big|_{z=e^{j\omega}} = X(e^{j\omega}) : DTFT \tag{2}$$

The DTFT converges if the ROC for the z-transform includes the unit circle $z = e^{j\omega}$:

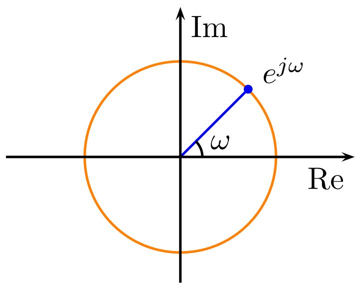


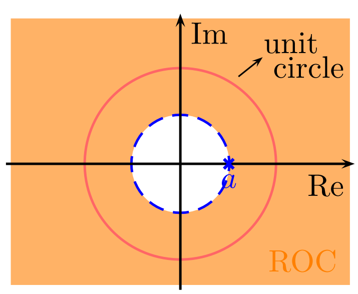
Figure 1: Berkeley EECS emeritus professor Lotfi Zadeh (above) and former professor Eliahu Jury (below) were among those who developed the theory of z transforms in the 1950s. Research in sampled systems was in part motivated by radar which came to prominence during World War II.

Example 1: $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} \quad \text{if } \underbrace{|az^{-1}| < 1}_{ROC:\ |z|>|a|}$$

DTFT:

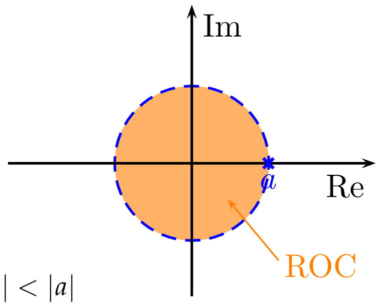
$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} \quad \text{if } |a| < 1$$



Poles and zeros: $X(z) = \frac{z}{z-a} \rightarrow$ pole at $z = a$, zero at $z = 0$.

Example 2: $x[n] = -a^n u[-n-1]$

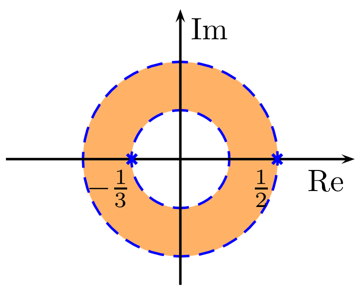
$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{n=1}^{\infty} -a^{-n} z^n \\ &= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n \\ &= 1 - \frac{1}{1-a^{-1}z} \quad \text{if } |a^{-1}z| < 1 \\ &= \frac{-a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}} \quad ROC: \ |z| < |a| \end{aligned}$$



DTFT converges is $|a| > 1$.

Example 3: $x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(\frac{-1}{3}\right)^n u[n]$

$$X(z) = \underbrace{\underset{|z|<\frac{1}{2}}{\frac{1}{1-\frac{1}{2}z^{-1}}} + \underset{|z|>\frac{1}{3}}{\frac{1}{1+\frac{1}{3}z^{-1}}}}_{ROC:\ \frac{1}{3}<|z|<\frac{1}{2}} = \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{2}} = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$



Example 4: $x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(\frac{-1}{3}\right)^n u[-n-1]$

$$ROC = \{z: \ |z| > \tfrac{1}{2}\} \cap \{z: \ |z| < \tfrac{1}{3}\} = \emptyset$$

Example 5: $x[n] = a^n$, $a \neq 0$.

$$x[n] = a^n u[n] + a^n u[-n-1]$$

$$ROC = \{z: \ |z| > a\} \cap \{z: \ |z| < a\} = \emptyset$$

Properties of the ROC

Section 10.2 *in* Oppenheim & Willsky

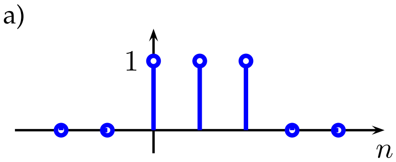
1) A ring or disk in the z-plane, centered at the origin.

2) ROC does not contain any poles.

3) For finite duration sequences, ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

Example 6:

a)

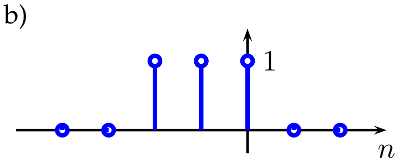


$X(z) = 1 + z^{-1} + z^{-2} = \frac{z^2+z+1}{z^2}$

ROC excludes $z = 0$ because of the pole at $z = 0$.

If $x[n] \neq 0$ for some $n > 0$, ROC excludes $z = 0$.

b)



$X(z) = z^2 + z + 1$

ROC excludes $z = \infty$.

If $x[n] \neq 0$ for some $n < 0$, ROC excludes $z = \infty$.

c) $x[n] = \delta[n] \rightarrow X(z) = 1$ and ROC is the entire z-plane, including $z = 0$ and $z = \infty$.

4) If $x[n]$ is right-sided ($x[n] \equiv 0 \; \forall n < N_1$, for some N_1) and if $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ are also in the ROC. (Ex 1)
$z = \infty$ included if $N_1 = 0$. (Ex 6)

5) If $x[n]$ is left-sided ($x[n] \equiv 0 \; \forall n > N_2$, for some N_2) and if $|z| = r_0$ is in the ROC, then $0 < |z| < r_0$ is also in the ROC. (Ex 2)

$z = 0$ included if $N_2 = 0$. (Ex 6)

6) If $x[n]$ is two-sided and if $|z| = r_0$ is in the ROC, then the ROC is a ring that includes $|z| = r_0$. (Ex 3)

The following hold when $X(z)$ is rational:

7) ROC is bounded by poles or extends to infinity.

8) If $x[n]$ is right-sided, ROC extends from the outermost pole to ∞.
$z = \infty$ included if $N_1 = 0$.
If $x[n]$ is left-sided, ROC extends from the innermost nonzero pole to 0.
$z = 0$ included if $N_2 = 0$.
For a two-sided sequence, the ROC is a ring:
Inner bound: Pole with largest magnitude that contributes to the right side.
Outer bound: Pole with smallest magnitude that contributes to the left side.
(Ex 3)

Inverse z-Transform by Partial Fraction Expansion

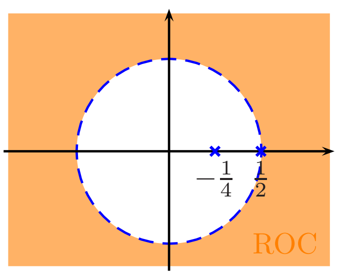
Example:

$$X(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}$$

$$A_1 = \left(1-\frac{1}{4}z^{-1}\right)X(z)\Big|_{z=\frac{1}{4}} = -1$$

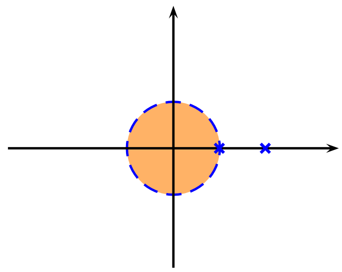
$$A_2 = \left(1-\frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}} = 2$$

1)



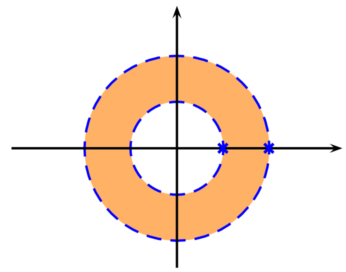
$$x[n] = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u[n]$$

2)



$$x[n] = -\left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u[-n-1]$$

3)



$$x[n] = -2\left(\frac{1}{2}\right)^n u[-n-1] - \left(\frac{1}{4}\right)^n u[n]$$

How to perform a PFE in general?

$$X(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}, \quad a_0 \neq 0$$

Suppose unrepeated poles: $d_1, d_2, \ldots, d_N$.

If $M < N$,

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

If $M \geq N$,

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}$$

$$\Downarrow$$

$$x[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \underbrace{\sum_{k=1}^{N} A_k d_k^n u[n]}_{\text{if right-sided}}$$

Example:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} \quad M = N = 2$$

$$= B_0 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

Matching coefficients: $A_1 = -9,\ A_2 = 8, B_0 = 2$.

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

Differentiation (in z-domain) Property:

$$x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \qquad ROC = R \qquad (3)$$

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz} \qquad ROC = R \qquad (4)$$

Proof:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[z]z^{-n} \\ \frac{dX(z)}{dz} &= \sum_{n=-\infty}^{\infty} -nx[n]z^{-(n+1)} = -z^{-1}\sum_{n=-\infty}^{\infty} nx[n]z^{-n} \end{aligned}$$

$$\sum_{n=-\infty}^{\infty} nx[n]z^{-n} = -z\frac{dX(z)}{dz}$$

Example:

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-az^{-1}}$$

$$na^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}\left\{\frac{1}{1-az^{-1}}\right\} = z\frac{az^{-2}}{(1-az^{-1})^2} = \frac{az^{-1}}{(1-az^{-1})^2}$$

Back to Partial Fraction Expansions: If d_k is a pole of multiplicity two, include two terms:

$$A_{k_1}\frac{1}{1-d_k z^{-1}} + A_{k_2}\frac{d_k z^{-1}}{(1-d_k z^{-1})^2}$$

$$\updownarrow$$

$$\left(A_{k_1} + A_{k_2} n\right) d_k^n u[n]$$

Example:

$$\begin{aligned} X(z) &= \frac{-\frac{1}{2}+z^{-1}}{(1-\frac{1}{2}z^{-1})^2} \qquad M=1,\ N=2 \qquad |z| > \frac{1}{2} \\ &= A_{11}\frac{1}{1-\frac{1}{2}z^{-1}} + A_{12}\frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \\ &= \frac{A_{11} + \frac{1}{2}\left(A_{12}-A_{11}\right)z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \end{aligned}$$

$$\left.\begin{aligned} A_{11} &= -\tfrac{1}{2} \\ \tfrac{1}{2}(A_{12}-A_{11}) &= 1 \end{aligned}\right\} A_{12} = \frac{3}{2}$$

$$x[n] = \left(-\frac{1}{2} + \frac{3}{2}n\right)\left(\frac{1}{2}\right)^n u[n]$$

Signal	Transform	ROC
$\delta[n]$	1	all z
$\delta[n-m]$	z^{-m}	all z except $z=0$ if $m>0$, all z except $z=\infty$ if $m<0$
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < a$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > a$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < a$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Table 1: z transforms of several functions.