EE120 - Fall'15 - Lecture 16 Notes[1]

Murat Arcak

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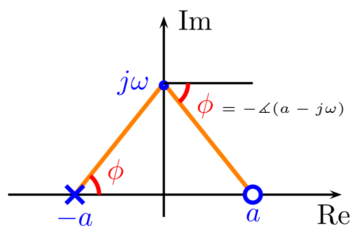
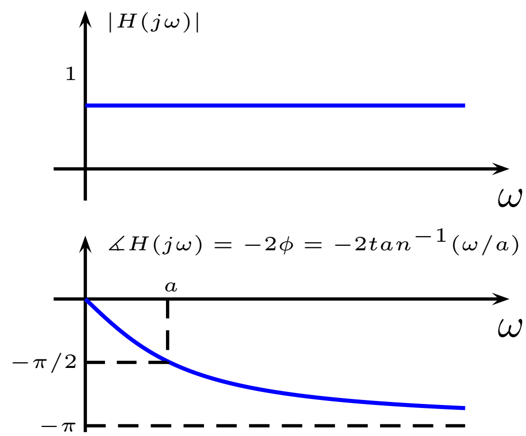
All-Pass Systems

Section 9.4.3 *in* Oppenheim & Willsky

What is the frequency response of an LTI system with transfer function:

$$H(s) = \frac{a-s}{s+a}, \quad a > 0?$$





General all-pass system:

$$H_{ap}(s) = \frac{\prod_{i=1}^{n}(a_i - s)}{\prod_{i=1}^{n}(s + a_i)}, \quad a_i > 0 \quad i = 1, ..., n. \tag{1}$$

For a stable and causal all-pass system, all zeros are in the right half-plane because they are mirror images of the poles.

Although $|H_{ap}(j\omega)| \equiv 1$, an all-pass system introduces delay:

$$e^{j\omega t} \rightarrow \boxed{H_{ap}(j\omega)} \rightarrow H_{ap}(j\omega)e^{j\omega t} = e^{j\measuredangle H_{ap}(j\omega)}e^{j\omega t} = e^{j\omega(t-\tau(\omega))}$$

where:

$$\tau(\omega) \triangleq -\frac{\measuredangle H_{ap}(j\omega)}{\omega} > 0.$$

Moreover, the system is not linear phase (i.e. $\tau(\omega)$ is not constant); therefore it causes phase distortion. (Recall Lecture 8.)

Minimum Phase Systems

A stable and causal LTI system is called *minimum phase* if all of its zeros are in the open left half-plane.

Any non-minimum phase transfer function $H(s)$ can be decomposed as:

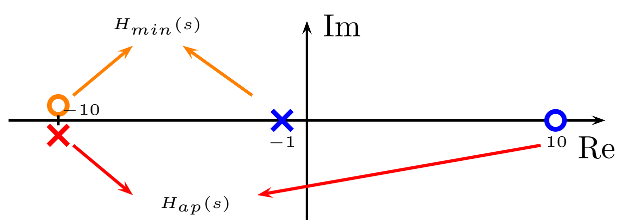
$$\boxed{H(s) = \underbrace{H_{min}(s)}_{\text{min-phase}} \underbrace{H_{ap}(s)}_{\text{all-pass}}} \tag{2}$$

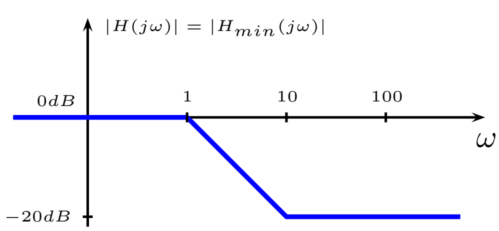
This decomposition explains the genesis of the term *minimum phase*:

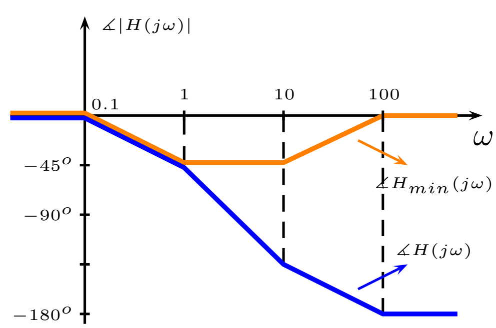
$$|H(j\omega)| = |H_{min}(j\omega)| \quad \text{since} \quad |H_{ap}(j\omega)| \equiv 1,$$

but the all-pass component adds more delay. Therefore, $H(s)$ and $H_{min}(s)$ have identical frequency responses in magnitude, but $H_{min}(s)$ has the minimum phase delay.

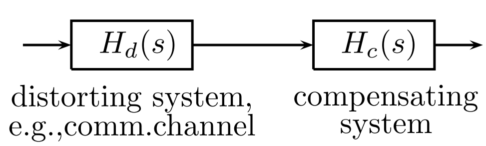
<u>Example:</u>
$$H(s) = \frac{10-s}{10(s+1)} = \underbrace{\frac{s+10}{10(s+1)}}_{H_{min}(s)} \cdot \underbrace{\frac{10-s}{s+10}}_{H_{ap}(s)}$$







<u>Example:</u>

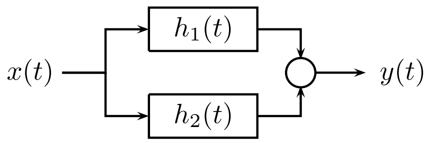


If $H_d(s)$ is minimum phase, we can simply choose $H_c(s) = \frac{1}{H_d(s)}$. If $H_d(s)$ is nonminimum phase, $H_c(s) = \frac{1}{H_d(s)}$ is unstable. To avoid instability, decompose: $H_d(s) = H_{d,min}(s)H_{d,ap}(s)$ and select:

$$H_c(s) = \frac{1}{H_{d,min}(s)} \quad \Longrightarrow \quad \underbrace{H_d(s)H_c(s) = H_{d,ap}(s)}_{\text{magnitude distortion eliminated}}$$

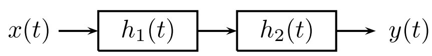
Transfer Functions of Interconnected LTI Systems

Section 9.8 *in* Oppenheim & Willsky



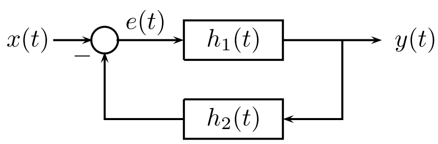
$$h(t) = h_1(t) + h_2(t)$$

$$\boxed{H(s) = H_1(s) + H_2(s)}$$



$$h(t) = h_1(t) * h_2(t)$$

$$\boxed{H(s) = H_1(s)H_2(s)}$$

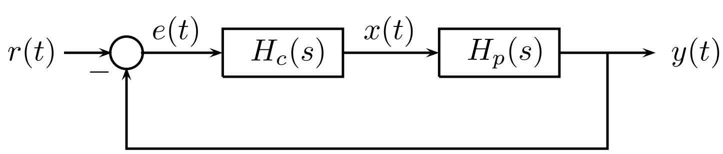


$$\begin{aligned} E(s) &= X(s) - H_2(s)Y(s) \\ Y(s) &= H_1(s)E(s) \\ &= H_1(s)X(s) - H_1(s)H_2(s)Y(s) \end{aligned}$$

$$(1 + H_1(s)H_2(s))Y(s) = H_1(s)X(s)$$

$$\boxed{\frac{Y(s)}{X(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}}$$

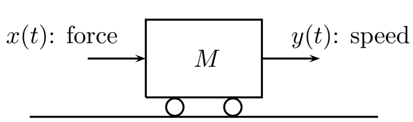
<u>Example:</u> Feedback Control



$r(t)$: reference signal to be tracked by $y(t)$
$H_c(s)$: controller, $H_p(s)$: system to be controlled - "plant"

$$H(s) = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)}$$

<u>Example:</u>

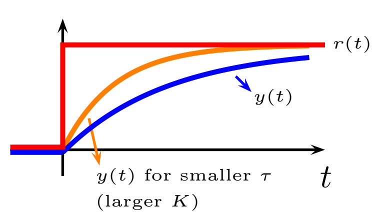


$$M\frac{dy}{dt} = x(t) \longrightarrow MsY(s) = X(s)$$

$$H_p(s) = \frac{1}{Ms}$$

Take constant gain controller: $H_c(s) = K$

$$H(s) = \frac{\frac{K}{Ms}}{1 + \frac{K}{Ms}} = \frac{1}{\tau s + 1} \qquad \tau = \frac{M}{K}$$



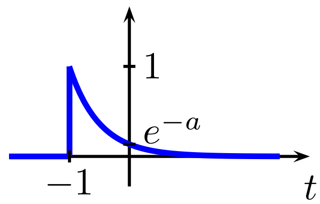
The Unilateral Laplace Transform

Section 9.9 *in* Oppenheim & Willsky

$$\mathcal{X}(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt \tag{3}$$

Identical to the bilateral Laplace transform if $x(t) = 0$ for $t < 0$.

Example: $x(t) = e^{-a(t+1)}u(t+1)$



$$X(s) = \frac{e^s}{s+a} \qquad Re\{s\} > -a$$
$$\mathcal{X}(s) = \frac{e^{-a}}{s+a} \qquad Re\{s\} > -a$$

Properties of the unilateral Laplace transform

Most properties of the bilateral Laplace transform also hold for the unilateral Laplace transform.

Exceptions:

Convolution:

$$x_1(t) * x_2(t) \longleftrightarrow \mathcal{X}_1(s)\mathcal{X}_2(s) \quad \text{if } x_1(t) = x_2(t) = 0 \text{ for all } t < 0$$

This follows from the convolution property of the bilateral Laplace transform which coincides with the unilateral transform because $x_1(t) = x_2(t) = 0,\ t < 0$.

Differentiation in Time:

$$\frac{dx(t)}{dt} \longleftrightarrow s\mathcal{X}(s) - x(0^-)$$

Repeated application gives:

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left\{\frac{dx(t)}{dt}\right\} \longleftrightarrow s\left(s\mathcal{X}(s) - x(0^-)\right) - \frac{dx}{dt}(0^-)$$
$$= s^2\mathcal{X}(s) - sx(0^-) - \frac{dx}{dt}(0^-)$$
$$\frac{d^3x(t)}{dt^3} = \frac{d}{dt}\left\{\frac{d^2x(t)}{dt^2}\right\} \longleftrightarrow s\left(s^2\mathcal{X}(s) - sx(0^-) - \frac{dx}{dt}(0^-)\right) - \frac{d^2x}{dt^2}(0^-)$$
$$= s^3\mathcal{X}(s) - s^2x(0^-) - s\frac{dx}{dt}(0^-) - \frac{d^2x}{dt^2}(0^-)$$

Solving differential equations with the unilateral Laplace transform

Example:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy}{dt} + 2y(t) = e^t \quad t \geq 0 \tag{4}$$

Initial condition $y(0^-) = a$, $\frac{dy}{dt}(0^-) = b$.

$$(s^2Y(s) - as - b) + 3(sY(s) - a) + 2Y(s) = \tfrac{1}{s-1}$$

$$(s^2 + 3s + 2)Y(s) = as + b + 3a + \tfrac{1}{s-1} = \tfrac{as^2+(b+2a)s+(1-b-3a)}{s-1}$$

$$Y(s) = \tfrac{as^2+(b+2a)s+(1-b-3a)}{(s+1)(s+2)(s-1)}$$

Partial fraction expansion:

$$\begin{aligned} Y(s) &= \frac{A_1}{s+1} + \frac{A_2}{s+2} + \frac{B}{s-1} \\ &= \frac{(A_1 + A_2 + B)s^2 + (A_1 + 3B)s + (2B - 2A_1 - A_2)}{(s+1)(s+2)(s-1)} \end{aligned}$$

Match coefficients:

$$\left.\begin{aligned} A_1 + A_2 + B &= a \\ A_1 + 3B &= b + 2a \\ 2B - 2A_1 - A_2 &= 1 - b - 3b \end{aligned}\right\} \quad \begin{aligned} B &= 1/6 \\ A_1 &= -\tfrac{1}{2} + 2a + b \\ A_2 &= \tfrac{1}{3} - a - b \end{aligned}$$

Then,

$$y(t) = \frac{1}{6}e^t + \left(-\frac{1}{2} + 2a + b\right)e^{-t} + \left(\frac{1}{3} - a - b\right)e^{-2t} \quad t \geq 0.$$

Compare this to the standard method for solving linear constant coefficient differential equations:

The first term in $y(t)$ above is the particular solution. If we substitute $y_p(t) = \frac{1}{6}e^t$ in (4):

$$\frac{d^2y_p(t)}{dt^2} + 3\frac{dy_p}{dt} + 2y_p(t) = e^t.$$

The second and third terms constitute the homogeneous solution. If we substitute $y_h(t) = A_1e^{-t} + A_2e^{-2t}$:

$$\frac{d^2y_h(t)}{dt^2} + 3\frac{dy_h}{dt} + 2y_h(t) = 0.$$

Thus, $y(t) = y_p(t) + y_h(t)$ and A_1 and A_2 are selected to satisfy the initial conditions.