EE120 - Fall'15 - Lecture 15 Notes[1]

Murat Arcak

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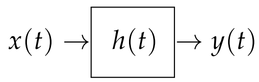
Analysis of LTI Systems using the Laplace Transform

Section 9.7 *in* Oppenheim & Willsky



$$Y(s) = H(s)X(s)$$

Causality: $h(t) = 0 \ \forall t < 0$

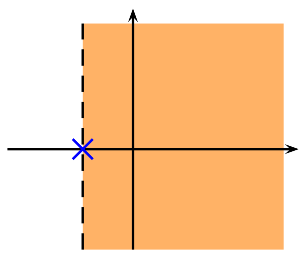
Stability: $\int_{-\infty}^{\infty} |h(t)|dt < \infty$

Determining Causality and Stability from $H(s)$

Causality: If $H(s)$ is rational, causality is equivalent to the ROC being the half plane to the right of the rightmost pole.

Example: $h(t) = e^{-t}u(t)$

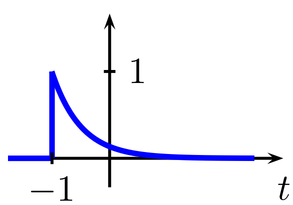
$$H(s) = \frac{1}{s+1} \quad Re\{s\} > -1$$



Example (why rationality of $H(s)$ matters):

$$h(t) = e^{-(t+1)}u(t+1)$$

(right-sided but not causal)



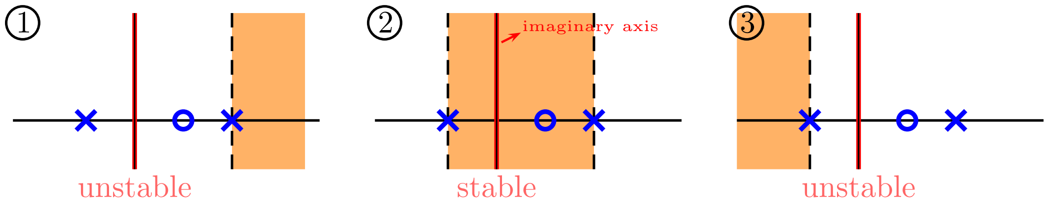
$$H(s) = \frac{e^s}{s+1} \quad Re\{s\} > -1 \longrightarrow$$ Not rational. If you don't check for rationality first, you can falsely conclude causality from the ROC.

Stability: An LTI system is stable if and only if the ROC of $H(s)$ includes the imaginary axis.

Example:

$$H(s) = \frac{s-1}{(s+1)(s-2)} = \frac{2/3}{s+1} + \frac{1/3}{s-2}$$

Possibilities for ROC:



Note that the same conclusion can be reached by applying the absolute integrability test to $h(t)$:

1. $h(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(t)$ — not absolutely integrable
2. $h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)$ — absolutely integrable
3. $h(t) = -\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(-t)$ — not absolutely integrable

Simpler stability test with additional causality assumption:

> A causal LTI system with rational $H(s)$ is stable if and only if all poles of $H(s)$ are in the open left half-plane, *i.e.*, all poles have negative real parts.

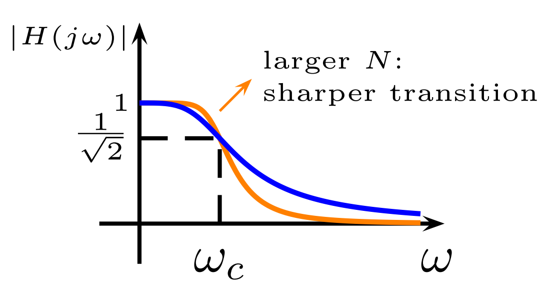
Note: "Open" left half-plane means that the imaginary axis is excluded.
Example (poles on the imaginary axis cause instability):

$$H(s) = \frac{1}{s} \quad \text{(integrator)}$$

If the input is $x(t) = u(t)$, then $X(s) = \frac{1}{s}$ and $Y(s) = H(s)X(s) = \frac{1}{s^2}$. Then, $y(t) = tu(t)$ which is unbounded although the input $x(t)$ is bounded.

Example (Butterworth filters):

$$|H(j\omega)|^2 = \frac{1}{1 + (\omega/\omega_c)^{2N}} \qquad \omega_c: \text{ cutoff frequency}, N: \text{ filter order}$$

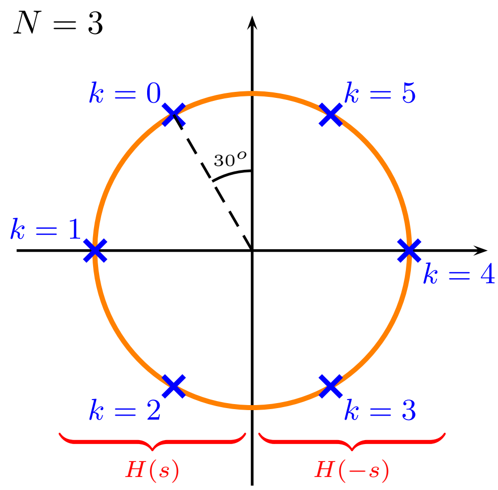


Derive the transfer function of a causal and stable LTI system with real-valued $h(t)$ that gives this frequency response.

$$|H(j\omega)|^2 = H(j\omega)\underbrace{H^*(j\omega)}_{\substack{H(-j\omega) \\ \text{since } h(t) \text{ is real}}} = \frac{1}{1+\left(\frac{j\omega}{j\omega_c}\right)^{2N}} \implies H(s)H(-s) = \frac{1}{1+\left(\frac{s}{j\omega_c}\right)^{2N}}$$

Thus, the roots of $\underbrace{1+\left(\frac{s}{j\omega_c}\right)^{2N}}=0$ are the poles of $H(s)$ combined with the poles of $H(-s)$.

$$\frac{s}{j\omega_c} = e^{j\left(\frac{\pi}{2N}+k\frac{2\pi}{2N}\right)} \quad k=0,1,...,2N-1$$
$$s = \underbrace{e^{j\frac{\pi}{2}}}_{j}\omega_c e^{j\left(\frac{\pi}{2N}+k\frac{2\pi}{2N}\right)}$$



Since the filter is to be causal and stable, $H(s)$ must contain the N poles in the left-half plane ($k = 0,1,...,N-1$) and $H(-s)$ must contain the rest $k = N,...,2N-1$.

Denominator of $H(s)$ for $N=3$:

$$(s+\omega_c)\underbrace{(s+\omega_c e^{j\frac{\pi}{3}})(s+\omega_c e^{-j\frac{\pi}{3}})}_{s^2+\underbrace{2cos(\frac{\pi}{3})}_{=1}\omega_c s+\omega_c^2}$$
$$= (s+\omega_c)(s^2+\omega_c s+\omega_c^2) = s^3+2\omega_c s^2+2\omega_c^2 s+\omega_c^3$$

Therefore, $H(s) = \frac{\omega_c^3}{s^3+2\omega_c s^2+2\omega_c^2 s+\omega_c^3}$ (so that $H(0) = \text{dc-gain} = 1$)

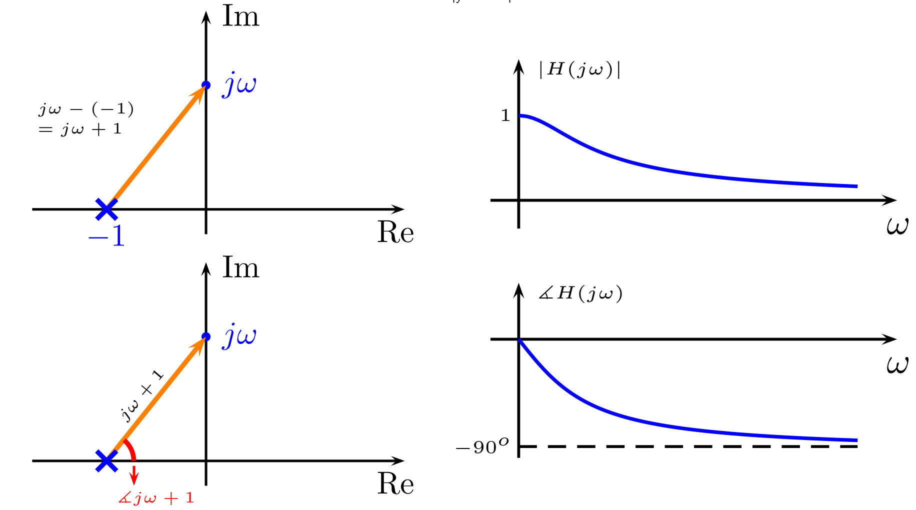
Normalized transfer function for the $N=3$ example above:

$$H^0(s) = \frac{1}{s^3+2s^2+2s+1} \qquad H(s) = H^0\left(\frac{s}{\omega_c}\right) \quad \text{for any desired } \omega_c$$

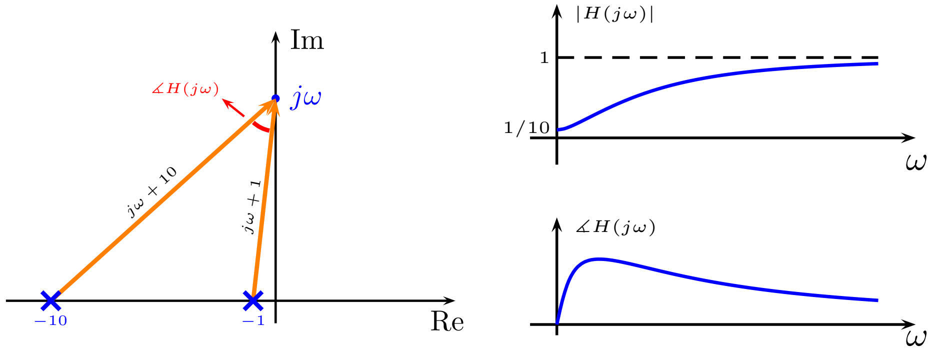
Evaluating the Frequency Response from the Pole-Zero Plot

Section 9.4 *in* Oppenheim & Willsky

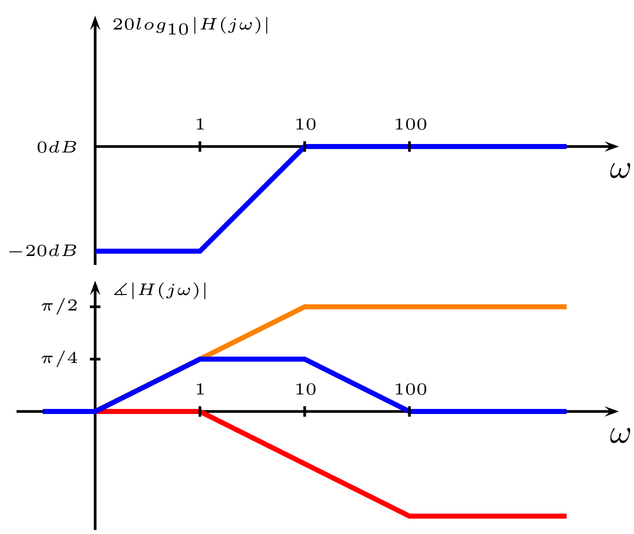
Example: $H(s) = \frac{1}{s+1}$ $\quad |H(j\omega)| = \frac{1}{|j\omega+1|}$



Example: $H(s) = \frac{s+1}{s+10}$



compare to Bode plots: $H(j\omega) = \frac{1}{10}\frac{1+j\omega}{1+j\omega/10}$



Example (second order system):

$$\begin{aligned} H(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ H(j\omega) &= \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} \end{aligned} \quad (1)$$

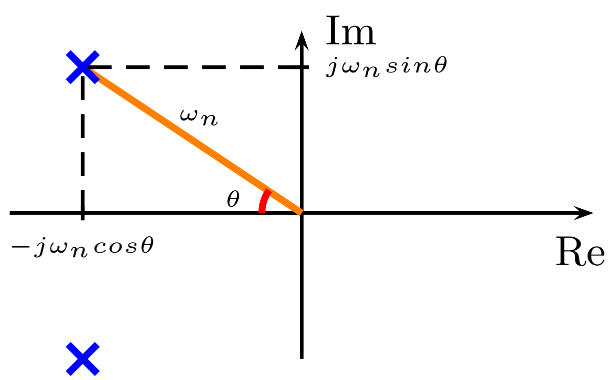
ζ : damping ratio, ω_n : natural frequency

Recall: resonance occurs if $\zeta < \frac{1}{\sqrt{2}} \approx 0.7$

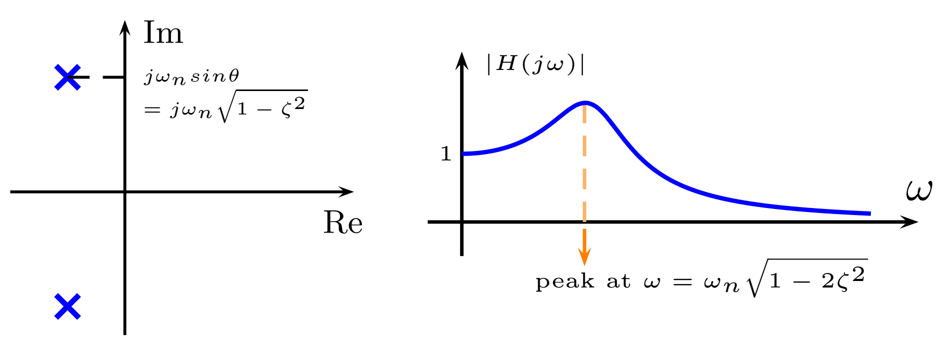
Poles of $H(s)$: $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, or $\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1 = 0$.
Then, $\frac{s}{\omega_n} = -\zeta \mp \sqrt{\zeta^2 - 1}$

Therefore, complex conjugate poles if $\zeta < 1$:

$$s_{1,2} = \omega_n(-cos(\theta) \mp jsin(\theta)) \text{ where } \theta \text{ defined by } \boxed{cos\theta = \zeta}$$



Resonance condition $\zeta < \frac{1}{\sqrt{2}}$ means $\theta > 45^o$



See Figure 1 below which we discussed in Lecture 5.

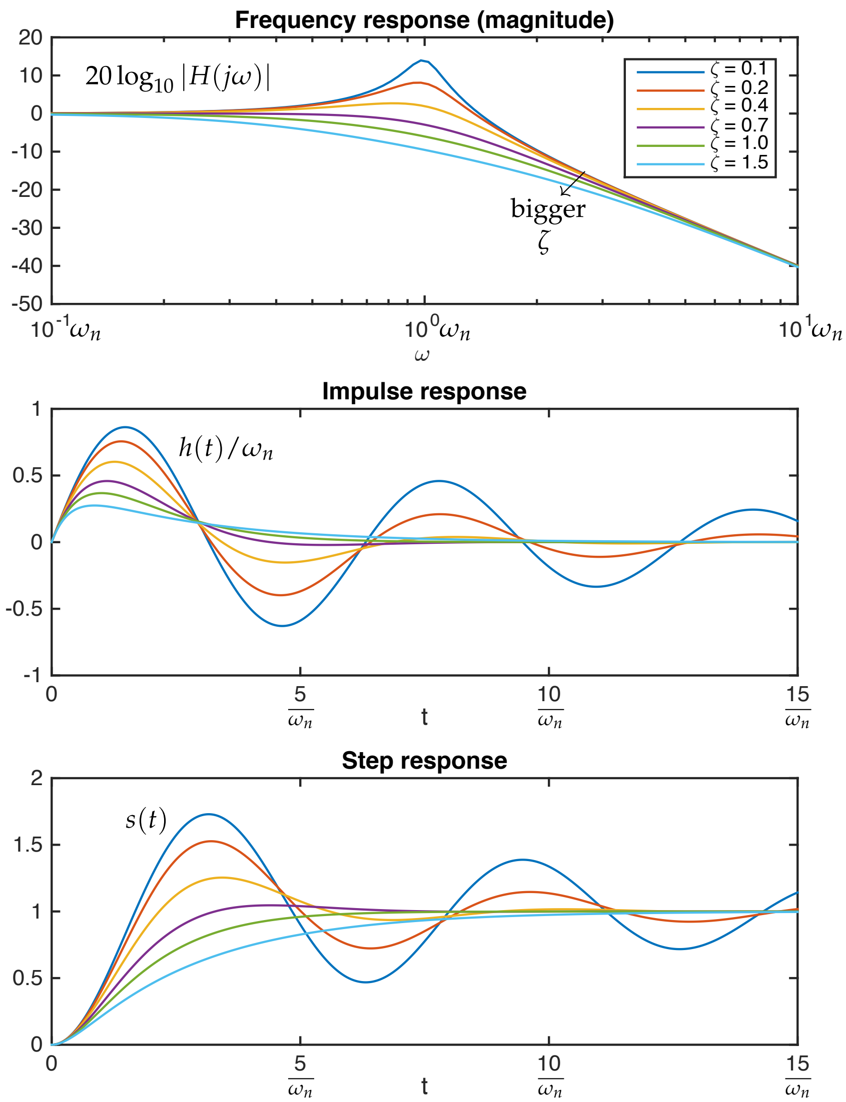


Figure 1: The frequency, impulse, and step responses for the second order system (1). Note from the frequency response (top) that a resonance peak occurs when $\zeta < 0.7$.