

EE120 - Fall'15 - Lecture 14 Notes¹

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The Laplace Transform continued

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

If the ROC contains the imaginary axis, then the Fourier transform is obtained by setting $s = j\omega$ in the Laplace transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt.$$

Properties of the Laplace Transform

Section 9.5 in Oppenheim & Willsky

Assume that $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ with $\text{ROC} = R$.

Linearity:

$$ax_1(t) + bx_2(t) \xleftrightarrow{\mathcal{L}} aX_1(s) + bX_2(s) \quad (1)$$

ROC contains $R_1 \cap R_2$, but can be larger: e.g., if $x_1(t) = x_2(t)$ and $a = -b$, then $ax_1(t) + bx_2(t) \equiv 0$ and ROC is the entire complex plane.

Time-Shift:

$$x(t - t_0) \leftrightarrow e^{-st_0} X(s) \quad (2)$$

ROC unchanged because:

$$\int_{-\infty}^{\infty} \underbrace{x(t - t_0)}_{\triangleq \tau} e^{-st} dt = \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} e^{-st_0} d\tau = \underbrace{e^{-st_0}}_{\substack{\text{this factor} \\ \text{doesn't change} \\ \text{convergence}}} \underbrace{\int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau}_{X(s)}$$

Shifting in the s-Domain:

$$e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s - s_0) \quad \text{ROC} = R + \text{Re}\{s_0\} \quad (3)$$

Compare to: $e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(j(\omega - \omega_0))$ in Fourier transforms.

Time-Scaling:

$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad \text{ROC} = a \cdot R \quad (4)$$

In particular, $x(-t) \xleftrightarrow{\mathcal{L}} X(-s)$, with $\text{ROC} = -R$.

Example:

$$\begin{aligned} \cos(\omega_0 t)u(t) &\xleftrightarrow{\mathcal{L}} \frac{s}{s^2 + \omega_0^2} & \text{Re}\{s\} > 0 \\ \cos(-\omega_0 t)u(-t) = \cos(\omega_0 t)u(-t) &\xleftrightarrow{\mathcal{L}} \frac{-s}{s^2 + \omega_0^2} & \text{Re}\{s\} < 0 \end{aligned}$$

Conjugation:

$$x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*) \quad \text{ROC unchanged} \quad (5)$$

Therefore, if $x(t)$ is real: $X(s) = X^*(s^*)$

Equivalent property in Fourier transforms: $x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)$

Convolution:

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)X_2(s) \quad \text{ROC contains } R_1 \cap R_2 \quad (6)$$

Differentiation in Time Domain:

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} sX(s) \quad \text{ROC contains } R \text{ but can be larger} \quad (7)$$

Example:

$$\begin{aligned} x(t) = u(t) &\xleftrightarrow{\mathcal{L}} X(s) = \frac{1}{s} & R = \{s : \text{Re}\{s\} > 0\} \\ \frac{dx(t)}{dt} = \delta(t) &\xleftrightarrow{\mathcal{L}} 1 \text{ for all } s & \text{ROC : entire complex plane} \end{aligned}$$

Example:

$$\begin{aligned} x(t) = \sin(\omega_0 t)u(t) &\xleftrightarrow{\mathcal{L}} \frac{\omega_0}{s^2 + \omega_0^2} & \text{Re}\{s\} > 0 \\ \frac{dx(t)}{dt} = \omega_0 \cos(\omega_0 t)u(t) &\xleftrightarrow{\mathcal{L}} \frac{\omega_0 s}{s^2 + \omega_0^2} & \text{Re}\{s\} > 0 \end{aligned}$$

Differentiation in the s-Domain:

$$-tx(t) \xleftrightarrow{\mathcal{L}} \frac{dX(s)}{ds} \quad \text{ROC unchanged for exponential signals} \quad (8)$$

Proof: $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$ then $\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} -tx(t)e^{-st} dt$.

Example:

$$\begin{aligned}
 e^{-at}u(t) &\xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \\
 te^{-at}u(t) &\xleftrightarrow{\mathcal{L}} -\frac{d}{ds} \left(\frac{1}{s+a} \right) = \frac{1}{(s+a)^2} \\
 t^2e^{-at}u(t) &\xleftrightarrow{\mathcal{L}} -\frac{d}{ds} \left(\frac{1}{(s+a)^2} \right) = \frac{2}{(s+a)^3} \\
 &\vdots \\
 t^n e^{-at}u(t) &\xleftrightarrow{\mathcal{L}} \frac{n!}{(s+a)^{n+1}}
 \end{aligned}$$

with $\text{Re}\{s\} > -a$ for all cases.

Special case $a = 0$: $u(t) \leftrightarrow \frac{1}{s}$, $tu(t) \leftrightarrow \frac{1}{s^2}$, ..., $t^n u(t) \leftrightarrow \frac{n!}{s^{n+1}}$

Example: Partial fraction expansion for repeated poles

Given $\text{ROC} = \{s : \text{Re}\{s\} > -1\}$, find the inverse Laplace transform for:

$$X(s) = \frac{1}{(s+1)(s+2)^2}$$

$$\begin{aligned}
 X(s) &= \frac{1}{(s+1)(s+2)^2} = \frac{A_1}{s+1} + \frac{A_{21}}{s+2} + \frac{A_{22}}{(s+2)^2} \\
 &= \frac{A_1(s+2)^2 + A_{21}(s+1)(s+2) + A_{22}(s+1)}{(s+1)(s+2)^2}
 \end{aligned}$$

$$\begin{aligned}
 \underbrace{(A_1 + A_{21})}_{=0} s^2 + \underbrace{(4A_1 + 3A_{21} + A_{22})}_{=0} s + \underbrace{(4A_1 + 2A_{21} + A_{22})}_{=1} &= 1 \\
 \implies A_1 = 1, \quad A_{21} = A_{22} = -1
 \end{aligned}$$

$$X(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2} \leftrightarrow x(t) = (e^{-t} - e^{-2t} - te^{-2t})u(t)$$

Integration in Time:

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s} X(s) \quad \text{ROC contains } R \cap \{s : \text{Re}\{s\} > 0\} \quad (9)$$

Follows from the convolution property: $\int_{-\infty}^t x(\tau) d\tau = x(t) * u(t)$.

Example: $x(t) = \delta(t) \leftrightarrow 1 \forall s$, $\int_{-\infty}^t x(\tau) d\tau = u(t) \leftrightarrow \frac{1}{s} \text{Re}\{s\} > 0$.

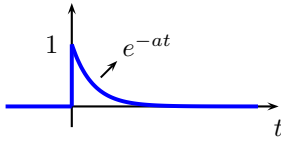
Initial Value Theorem:

If $x(t) = 0$ for all $t < 0$ and contains no impulses or singularities at $t = 0$,

$$x(0^+) = \lim_{s \rightarrow \infty} sX(s) \quad (10)$$

Example:

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}$$

$$\lim_{s \rightarrow \infty} \frac{s}{s+a} = 1 = e^{-at}u(t)|_{t=0^+}$$


Final Value Theorem:

If $x(t) = 0$ for all $t < 0$ and $x(t)$ has a finite limit as $t \rightarrow \infty$, then

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s) \quad (11)$$

Transfer Functions of LTI Systems

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

From the convolution property:

$$\boxed{Y(s) = H(s)X(s)}$$

where $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt$ is called the "transfer function" or "system function."

Finding the transfer function from differential equations

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

Take the Laplace transform of both sides and use differentiation property:

$$\sum_{k=0}^N a_k s^k Y(s) = \sum_{k=0}^M b_k s^k X(s)$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^M b_k s^k}{\sum_{k=0}^N a_k s^k} = \frac{b_M s^M + b_{M-1} s^{M-1} + \dots + b_0}{a_N s^N + a_{N-1} s^{N-1} + \dots + a_0}}$$

Poles of the system: roots of $a_N s^N + a_{N-1} s^{N-1} + \dots + a_0$

Zeros of the system: roots of $b_M s^M + b_{M-1} s^{M-1} + \dots + b_0$

Section 9.7 in Oppenheim & Willsky

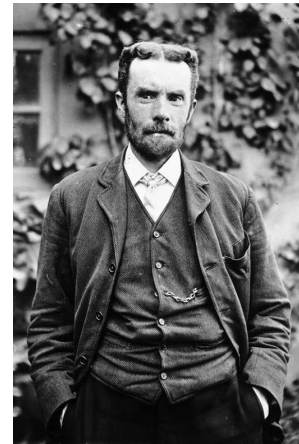
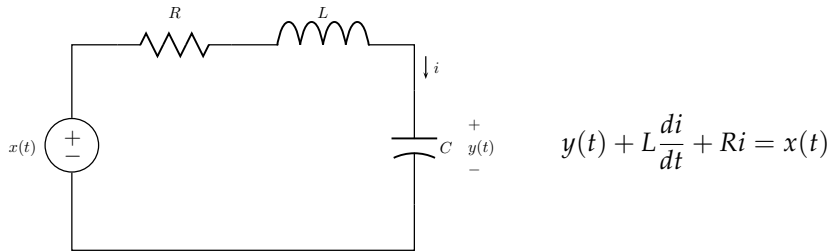


Figure 1: **Oliver Heaviside (1850-1925)**, a self-taught electrical engineer, invented the "operational calculus" where the differential operator $\frac{d}{dt}$ is treated as a symbol (' s ' in our case) and a linear differential equation is manipulated algebraically. Dynamic circuit elements could now be represented with simple algebraic expressions similar to Ohm's Law (e.g. Ls for inductance). Heaviside's method had found widespread use by the time others established the full mathematical justification with the help of a transform used by Laplace a century earlier. Heaviside had many other contributions, including condensing Maxwell's theory of electromagnetism into the four vector equations known today. He further coined the terms "inductance" and "impedance," and the unit step $u(t)$ is sometimes referred to as the Heaviside function.

Example:



Substitute $i = C \frac{dy}{dt}$:

$$LC \frac{d^2 y}{dt^2} + RC \frac{dy}{dt} + y(t) = x(t).$$

Therefore,

$$H(s) = \frac{1}{LCs^2 + RCs + 1}.$$

Poles: $\frac{-RC \pm \sqrt{R^2 C^2 - 4LC}}{2LC}$. No zeros.

How do poles affect the system response?

If there are no repeated poles, partial fraction expansion gives:

$$H(s) = \sum_{i=1}^N \frac{A_i}{s - \alpha_i} \quad (12)$$

where $\alpha_i, i = 1, \dots, N$, are the poles. Then, assuming causality:

$$h(t) = \sum_{i=1}^N A_i e^{\alpha_i t} u(t) \quad (13)$$

Each pole α_i contributes an exponential term $e^{\alpha_i t}$ to the response.²

If α_i is repeated m times, then the system response includes:

$$t^{m-1} e^{\alpha_i t}, \dots, t e^{\alpha_i t}, e^{\alpha_i t}$$

Example: In the RLC circuit above, we expect oscillatory response if

$$R^2 C^2 < 4LC.$$

How do zeros affect the system response?

Suppose $s = \beta$ is a zero of $H(s)$, i.e., $H(\beta) = 0$. Then:

$$e^{\beta t} \rightarrow \boxed{h(t)} \rightarrow y(t) = H(\beta) e^{\beta t} = 0$$

² See Figure 2 on the next page which we discussed in Lecture 2.

Thus, the zero $s = \beta$ blocks inputs of the form $e^{\beta t}$ from appearing at the output.

Example: Consider the RLC circuit above and redefine the output to be the current instead of the capacitor voltage: $y(t) \triangleq i(t)$. Then,

$$C \underbrace{\frac{d}{dt} \left(x(t) - Ry(t) - L \frac{dy}{dt} \right)}_{\text{voltage across capacitor}} = y(t).$$

Rearrange terms:

$$LC \frac{d^2 y}{dt^2} + RC \frac{dy}{dt} + y = C \frac{dx}{dt}.$$

Then, the transfer function is:

$$H(s) = \frac{Cs}{LCs^2 + RCs + 1}.$$

Zero at $s = 0$ blocks constant inputs: When $x(t) = e^{0t} \equiv 1$, $y(t) \equiv 0$.

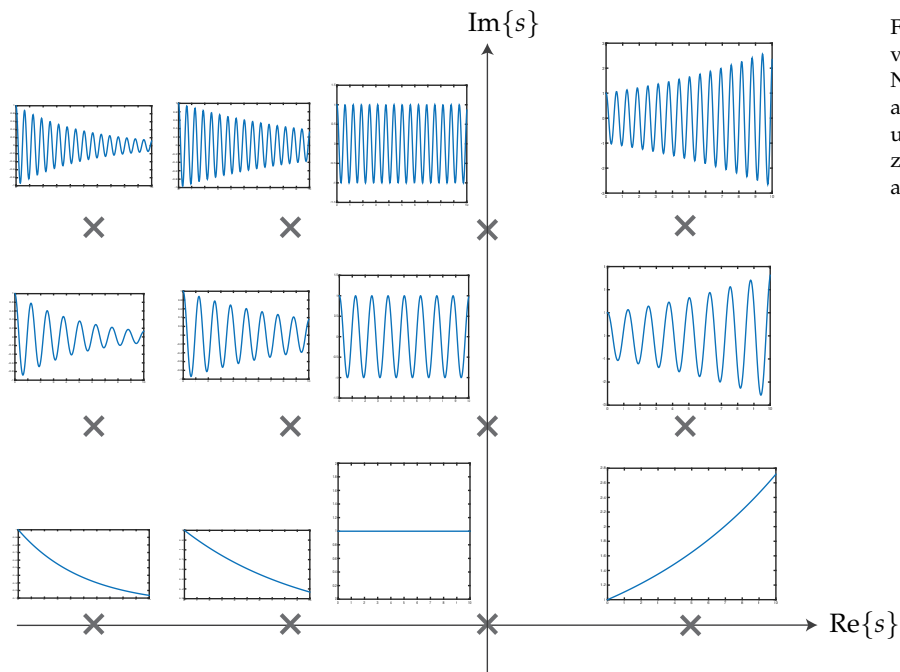


Figure 2: The real part of e^{st} for various values of s in the complex plane. Note that e^{st} is oscillatory when s has an imaginary component. It grows unbounded when $\text{Re}\{s\} > 0$, decays to zero when $\text{Re}\{s\} < 0$, and has constant amplitude when $\text{Re}\{s\} = 0$.