EE120 - Fall'15 - Lecture 13 Notes[1]

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The Laplace Transform

Chapter 9 *in* Oppenheim & Willsky

The Laplace transform of $x(t)$ is

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{1}$$

where s is a complex variable. $X(s)|_{s=j\omega}$ is the Fourier transform.

Recall from Lecture 2:

$$e^{st} \rightarrow \boxed{h(t)} \rightarrow H(s)e^{st} \quad \text{where} \quad H(s) \triangleq \int_{-\infty}^{\infty} h(t)e^{-st}dt.$$

Thus the transfer function $H(s)$ is the Laplace transform of $h(t)$.

Example 1:

$$\boxed{x(t) = e^{-at}u(t)} \leftrightarrow X(j\omega) = \frac{1}{j\omega + a} \quad \text{if} \quad a > 0 \quad \text{(Lecture 4)}$$

Find the Laplace transform:

$$X(s) = \int_{0}^{\infty} e^{-at}e^{-st}dt.$$

Let σ denote the real part of s ($s = \sigma + j\omega$):

$$\begin{aligned} X(s) &= \int_0^{\infty} e^{-(a+\sigma)t}e^{-j\omega t}dt = \underset{\text{(convergence if } a+\delta>0)}{\text{Fourier transform of } e^{-(a+\sigma)}u(t).} \\ &= \frac{1}{j\omega + (a+\sigma)} = \frac{1}{s+a} \end{aligned}$$

Therefore,

$$\boxed{X(s) = \frac{1}{s+a} \text{ if } \delta = Re\{s\} > -a}$$

If $a = 0$ (unit step), Fourier transform doesn't converge, but the Laplace transform does for $Re\{s\} > 0$.

Example 2:

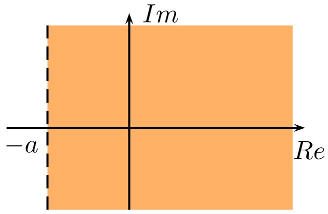
$$\boxed{x(t) = -e^{-at}u(-t)}$$

$$\begin{aligned} X(s) &= -\int_{-\infty}^{0} e^{-at}e^{-st}dt \quad (\text{ let } \tau = -t) \\ &= -\int_{\infty}^{0} e^{(a+s)\tau}(-d\tau) = -\int_0^{\infty} e^{(a+s)\tau}d\tau = -\frac{1}{a+s}e^{(a+s)\tau}|_0^{\infty} \end{aligned}$$

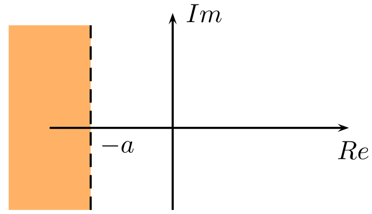
If $Re\{a+s\} < 0$, then $e^{(a+s)\tau} \rightarrow 0$ as $\tau \rightarrow \infty$: $\boxed{= \frac{1}{s+a} \text{ if } Re\{s\} < -a}$

Region of Convergence (ROC)

Ex. 1:



Ex. 2:



If the ROC includes the imaginary axis, then the Fourier transform converges.

Example 3: $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{(s+1)(s+2)} \qquad ROC = \{s|Re\{s\} > -2\} \cap \{s|Re\{s\} > -1\} = \{s|Re\{s\} > -1\}$$

Example 4: $x(t) = e^{-at}u(t)$, a: complex.

$$X(s) = \frac{1}{s+a} \text{ if } Re\{s+a\} > 0, \text{ i.e., if } Re\{s\} > -Re\{a\}.$$

Example 5: $cos(\omega_0 t)u(t) = \frac{1}{2}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-j\omega_0 t}u(t)$

From Example 4 the Laplace transform is:

$$\frac{1}{2}\underbrace{\frac{1}{s-j\omega_0}}_{Re\{s\}>0} + \frac{1}{2}\underbrace{\frac{1}{s+j\omega_0}}_{Re\{s\}>0} = \frac{1}{2}\frac{2s}{s^2+\omega_0^2} = \frac{s}{s^2+\omega_0^2}$$

$$\boxed{\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+\omega_0^2} \quad Re\{s\} > 0} \tag{2}$$

A similar derivation shows:

$$\boxed{\sin(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{s^2+\omega_0^2} \quad Re\{s\} > 0} \tag{3}$$

Example 6: $e^{-at}cos(\omega_0 t)u(t) = \frac{1}{2}e^{-(a-j\omega_0)t}u(t) + \frac{1}{2}e^{-(a+j\omega_0)t}u(t)$

From Example 4:

$$\frac{1}{2}\underbrace{\frac{1}{s+a-j\omega_0}}_{Re\{s\}>-a} + \frac{1}{2}\underbrace{\frac{1}{s+a+j\omega_0}}_{Re\{s\}>-a} = \frac{(s+a)}{(s+a)^2+\omega_0^2}$$

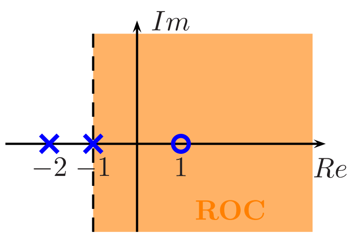
$$\boxed{e^{-at}\cos(\omega_0 t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{(s+a)}{(s+a)^2+\omega_0^2} \quad Re\{s\} > -a} \tag{4}$$

Poles and Zeros of Laplace Transforms

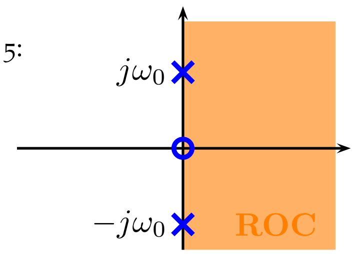
$$X(s) = \frac{N(s)}{D(s)} \tag{5}$$

Zeros: roots of $N(s)$ ($N(s) = 0$), poles: roots of $D(s)$ ($D(s) = 0$).

Ex. 3:



Ex. 5:



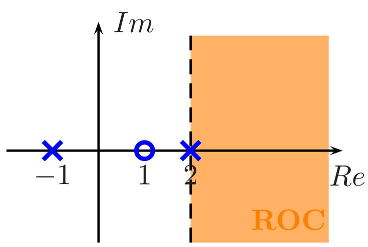
Zeros marked with "o", and poles with "x".

Example 7:

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$$

Laplace transform of $\delta(t)$: $\int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1$ for all $s \in \mathcal{C}$.

$$\begin{aligned} X(s) &= 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} \\ &= \frac{3(s+1)(s-2) - 4(s-2) + (s+1)}{3(s+1)(s-2)} = \frac{3s^2 - 6s + 3}{3(s+1)(s-2)} \\ &= \frac{s^2 - 2s + 1}{(s+1)(s-2)} = \frac{(s-1)^2}{(s+1)(s-2)} \quad \text{if} \quad Re\{s\} > 2. \end{aligned}$$



Note: In each example, the ROC excludes poles (Property 2 below).

Properties of the ROC

Section 9.2 *in* Oppenheim & Willsky

Note that the Laplace and Fourier transforms are related by

$$\mathcal{L}\{x(t)\} = \mathcal{F}\{x(t)e^{-\sigma t}\} \text{ where } \sigma = Re\{s\}.$$

The properties below characterize the ROC for the Laplace transform by examining the absolute integrability condition[2] for the convergence of the Fourier transform:

[2] The other Dirichlet conditions (Lecture 4) are assumed to be satisfied.

$$\int_{-\infty}^{\infty} |x(t)|e^{-\sigma t}dt < \infty. \tag{6}$$

1) ROC consists of strips parallel to the imaginary axis.

Justification: Since (6) depends only on $\sigma = Re\{s\}$, if a particular point is in the ROC then the entire vertical line with the same real part must be in the ROC.

2) For rational Laplace transforms (X(s)=N(s)/D(s)), ROC does not contain poles.

3) If $x(t)$ is of finite duration and absolutely integrable, then the ROC is the entire complex plane.

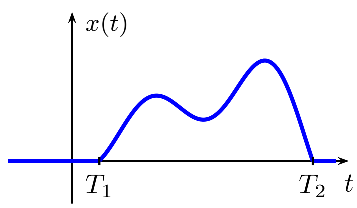
Justification: For a finite duration signal like the one below, condition (6) becomes:

$$\int_{T_1}^{T_2} |x(t)|e^{-\sigma t}dt < \infty.$$

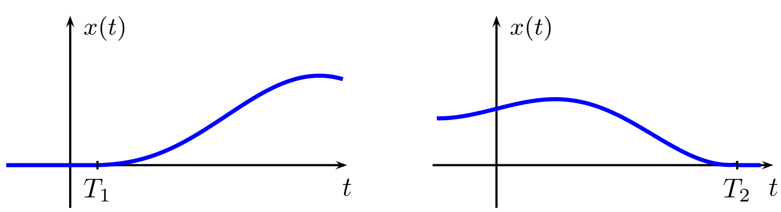
To see that this indeed holds for all σ, note that

$$\int_{T_1}^{T_2} |x(t)|e^{-\sigma t}dt \leq \max_{t\in[T_1,T_2]} e^{-\sigma t} \cdot \int_{T_1}^{T_2} |x(t)|dt$$

where $e^{-\sigma t}$ is bounded in the bounded interval $[T_1, T_2]$ and $\int_{T_1}^{T_2} |x(t)|dt$ is bounded from the absolute integrability of $x(t)$.



Definition: $x(t)$ is right-sided if $x(t) = 0$ prior to some finite time T_1, left-sided if $x(t) = 0$ after some finite time T_2, and two-sided if neither is the case.



4) For a right-sided $x(t)$, if ROC is not empty then it extends to $+\infty$ along the real axis. (Examples 1,3,4,5,6,7)

Justification: Suppose

$$\int_{T_1}^{\infty} |x(t)|e^{-\sigma_0 t}dt < \infty$$

for some σ_0, so that the vertical line $Re\{s\} = \sigma_0$ is in the ROC.
Then, for any $\sigma \geq \sigma_0$, $\int_{T_1}^{\infty} |x(t)|e^{-\sigma t}dt < \infty$ because:

$$\begin{aligned}\int_{T_1}^{\infty} |x(t)|e^{-\sigma t}dt &= \int_{T_1}^{\infty} |x(t)|e^{-\sigma_0 t}\underbrace{e^{-(\sigma-\sigma_0)t}}_{\leq e^{-(\sigma-\sigma_0)T_1} \quad t\geq T_1}dt \\ &\leq e^{-(\sigma-\sigma_0)T_1}\underbrace{\int_{-\infty}^{\infty} |x(t)|e^{-\sigma_0 t}dt}_{<\infty}\end{aligned}$$

5) For a left-sided x(t), if ROC is not empty then it extends to $-\infty$ along the real axis. (Example 2)

6) For a two-sided $x(t)$, if ROC is not empty then it is a vertical strip.

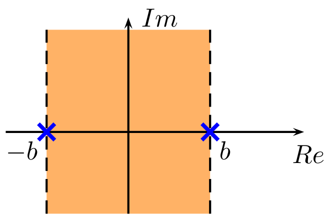
Example 8:

$$x(t) = e^{-b|t|} = \begin{cases} e^{-bt} & t \geq 0 \\ e^{bt} & t < 0 \end{cases}$$

$$e^{-bt}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b} \quad \text{if } Re\{s\} > -b \quad \text{(Example 1)} \qquad (7)$$

$$e^{bt}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{-1}{s-b} \quad \text{if } Re\{s\} < b \quad \text{(Example 2)} \qquad (8)$$

If $b \leq 0$, $ROC = \emptyset$. If $b > 0$, $ROC = \{s| -b < Re\{s\} < b\}$.



7) If $X(s)$ is rational, then ROC is bounded by poles or extends to $\mp\infty$.

8) If $X(s)$ is rational and $x(t)$ is right-sided, ROC is the half-plane to the right of the rightmost pole. (Examples 1,3,4,5,6,7)
If $x(t)$ is left-sided, ROC is the half-plane to the left of the leftmost pole. (Example 2)

Inverse Laplace Transform by Partial Fraction Expansion

Section 9.3 *in* Oppenheim & Willsky

Example 9:

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{(A+B)s + (2A+B)}{(s+1)(s+2)}$$

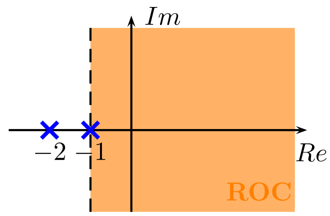
$$\left.\begin{array}{r} A+B=0 \\ 2A+B=1 \end{array}\right\} \begin{array}{l} A=1 \\ B=-1 \end{array}$$

Note that:

$$\frac{1}{s+1} \begin{array}{l} \rightarrow \quad e^{-t}u(t) \quad Re\{s\} > -1 \\ \searrow \quad -e^{-t}u(-t) \quad Re\{s\} < -1 \end{array} \qquad \frac{1}{s+2} \begin{array}{l} \rightarrow \quad e^{-2t}u(t) \quad Re\{s\} > -2 \\ \searrow \quad -e^{-2t}u(-t) \quad Re\{s\} < -2 \end{array}$$

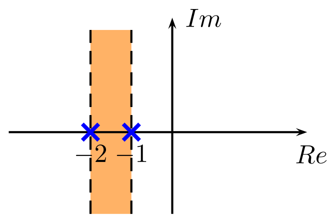
Thus, $x(t)$ can't be determined uniquely unless the ROC is specified.

Possibilities:
1) $x(t) = e^{-t}u(t) - e^{-2t}u(-t)$, if $Re\{s\} > -1$.

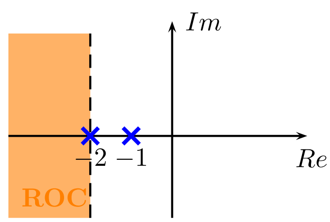


~~2)~~ $x(t) = e^{-t}u(t) - e^{-2t}u(-t)$, $ROC = \emptyset$
since $Re\{s\} > -1$ and $Re\{s\} < -2$ do not intersect.

3) $x(t) = -e^{-t}u(-t) + e^{-2t}u(t)$ if $-2 < Re\{s\} < -1$



4) $x(t) = -e^{-t}u(-t) - e^{-2t}u(-t)$ if $Re\{s\} < -2$



Signal	Transform	ROC
$\delta(t)$	1	all s
$u(t)$	$\frac{1}{s}$	$Re\{s\} > 0$
$-u(-t)$	$\frac{1}{s}$	$Re\{s\} < 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$Re\{s\} > 0$
$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$Re\{s\} < 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$Re\{s\} > -a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$Re\{s\} < -a$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$Re\{s\} > -a$
$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$Re\{s\} < -a$
$\delta(t-T)$	e^{-sT}	all s
$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$Re\{s\} > 0$
$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$Re\{s\} > 0$
$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$Re\{s\} > -a$
$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$Re\{s\} > -a$

Table 1: Laplace transforms of several functions.