EE120 - Fall'15 - Lecture 12 Notes[1]

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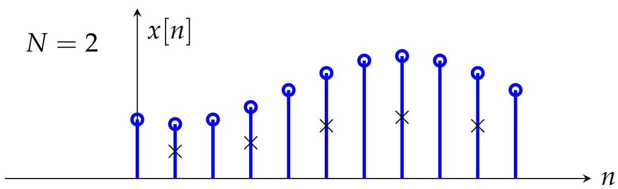
Sampling of Discrete-Time Signals

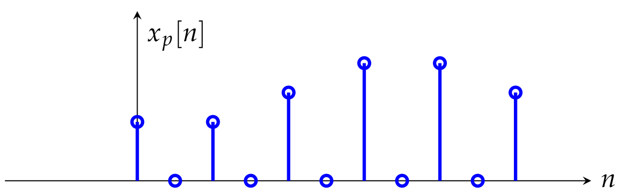
Section 7.5 *in* Oppenheim & Willsky

Impulse Train Sampling

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \tag{1}$$

$$x_p[n] = x[n]p[n] = \sum_{k=-\infty}^{\infty} x[kN]\delta[n - kN] \tag{2}$$





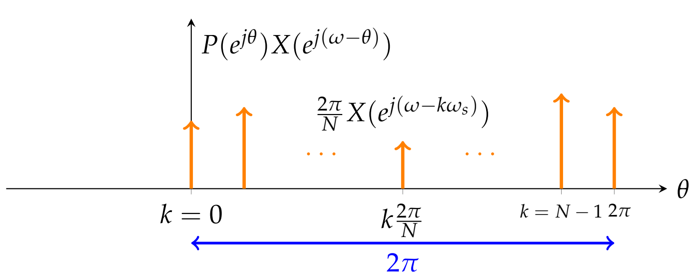
Fourier series coefficients of $p[n]$: $a_k = \frac{1}{N}$ for all k. Therefore,

$$P(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\Big(\omega - k \underbrace{\frac{2\pi}{N}}_{=\omega_s}\Big) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s).$$

Since $x_p[n] = x[n]p[n]$, the multiplication property of DTFT implies:

$$X_p(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} P(e^{j\theta}) X(e^{j(\omega-\theta)}) d\theta \tag{3}$$

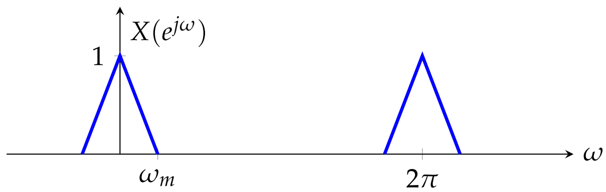
where the integrand has the form:

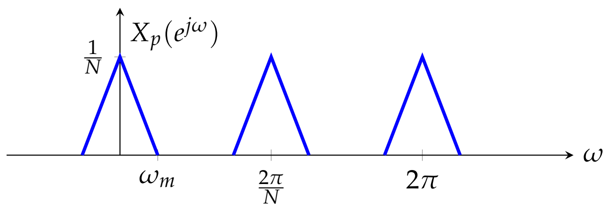


Thus,

$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)}) \quad \omega_s = \frac{2\pi}{N} \tag{4}$$

No aliasing if $\omega_s > 2\omega_m$ where ω_m is the bandwidth of $X(e^{j\omega})$.
Below is an example of $X(e^{j\omega})$ and $X_p(e^{j\omega})$ for $N = 2$:

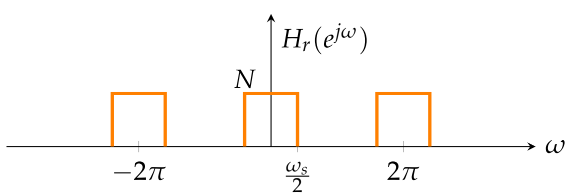




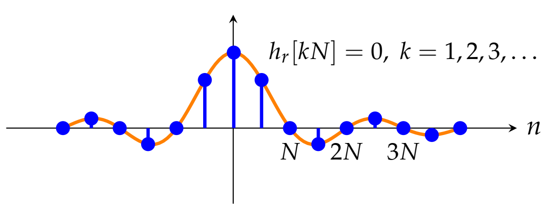
Compare (4) to impulse train sampling of CT signals:

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \quad \omega_s = \frac{2\pi}{T}. \tag{5}$$

Ideal Reconstruction Filter



Impulse response: $h_r[n] = \dfrac{\sin(\pi n/N)}{\pi n/N}$ when $n \neq 0$, and $h_r[0] = 1$

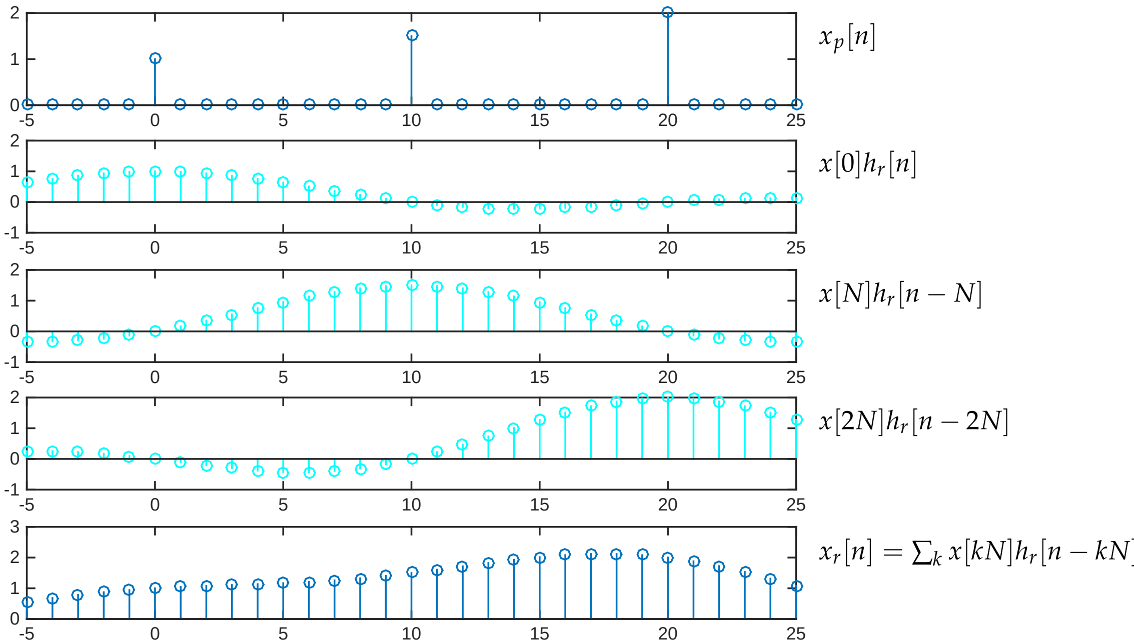


Convolution with $h_r[n]$ results in a "bandlimited interpolation:"

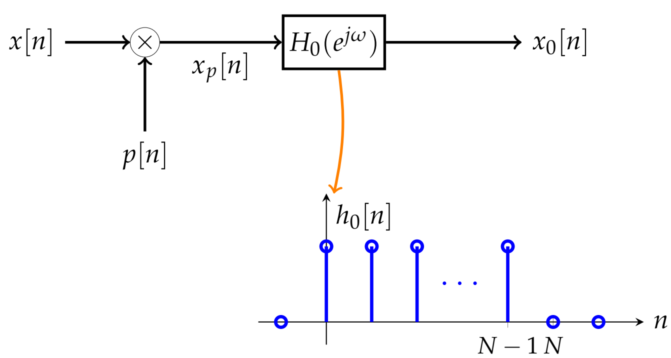
$$x_r[n] = x_p[n] * h_r[n] = \sum_{k=\infty}^{\infty} x[kN]\underbrace{\delta(n-kN) * h_r[n]}_{=h_r[n-kN]} \tag{6}$$

$$= \sum_{k=-\infty}^{\infty} x[kN]h_r[n-kN] \tag{7}$$

Bandlimited interpolation is illustrated below on a signal sampled with period $N = 10$.

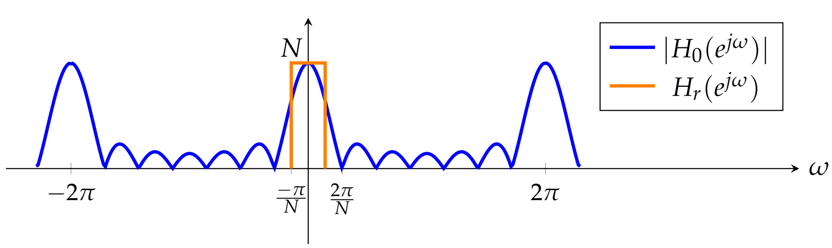


Zero-Order Hold Reconstruction



Taking the DTFT of this impulse response we get the frequency response:

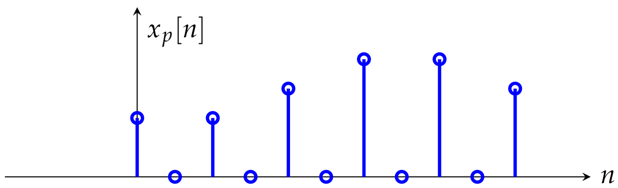
$$H_0(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)} \quad \omega \neq 0, \text{ and } H_0(e^{j0}) = N \tag{8}$$

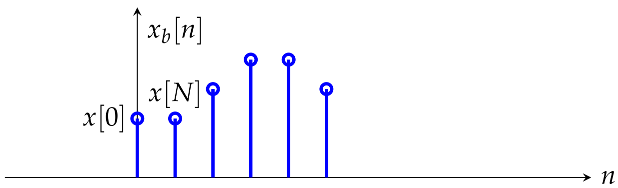


Downsampling

Select every Nth sample, discard the rest:

$$x_b[n] = x[Nn] = x_p[Nn]. \tag{9}$$

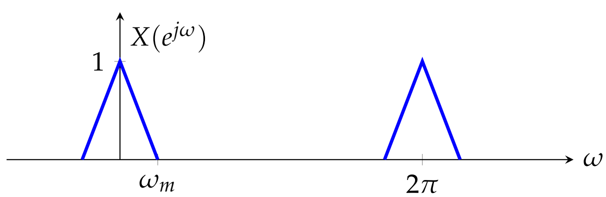


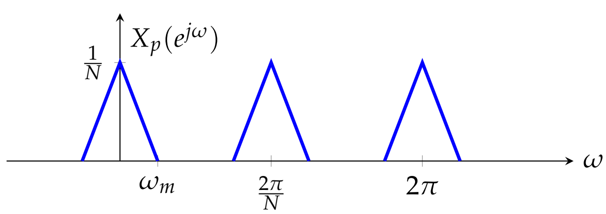


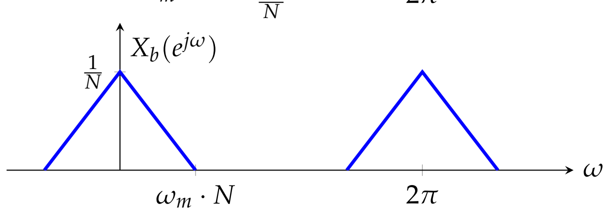
$$X_p(e^{j\omega}) = \ldots + x[0] + x[N]e^{-j\omega N} + x[2N]e^{-j\omega 2N} + \ldots \tag{10}$$

$$X_b(e^{j\omega}) = \ldots + x[0] + x[N]e^{-j\omega} + x[2N]e^{-j2\omega} + \ldots \tag{11}$$

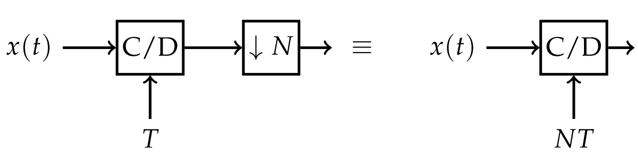
$$\Longrightarrow \quad \boxed{X_b(e^{j\omega}) = X_p(e^{j\omega/N})} \tag{12}$$





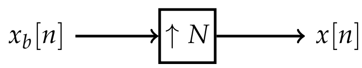


Note that sampling of a CT signal followed by downsampling is equivalent to sampling of the CT signal at a slower rate:

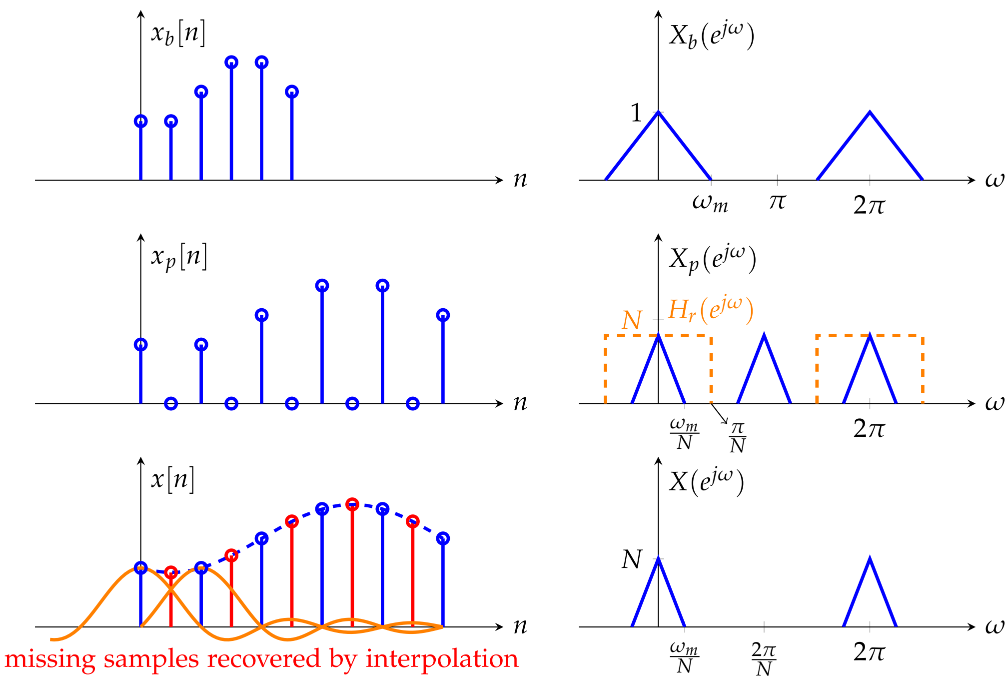


Upsampling

Inverse of downsampling:

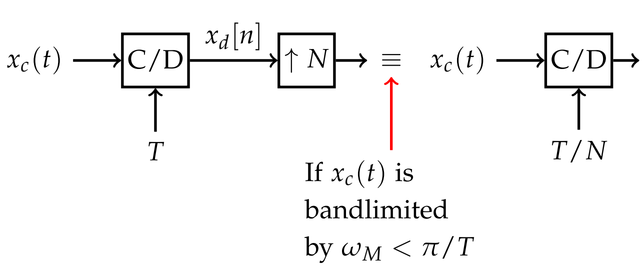


- Expand signal by N: $x_p[n] = \begin{cases} x_b[n/N] & n = 0, \mp N, \mp 2N, \dots \\ 0 & \text{otherwise} \end{cases}$
- Obtain $x[n]$ from $x_p[n]$ by interpolation (reconstruction filter)

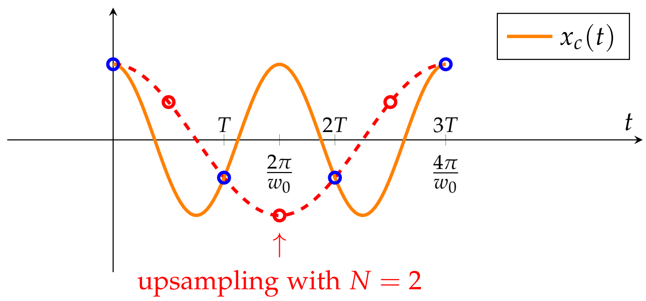


Is sampling a CT signal followed by upsampling equivalent to sampling the CT signal faster in the first place?

Not if the initial, slower sampling introduced aliasing. Yes, if the original sampling period T was small enough to avoid aliasing:

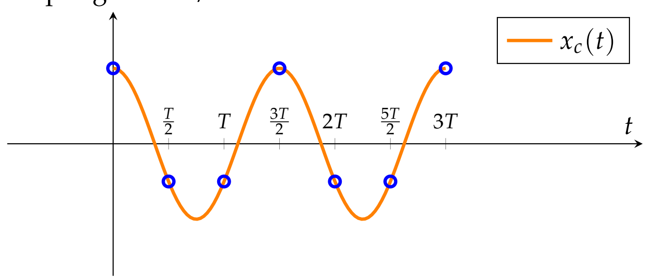


<u>Example:</u> $x_c(t) = \cos \omega_0 t \qquad \omega_s = \frac{3}{2}\omega_0$



Interpolated (red) samples don't match the results of $\times 2$ faster sampling.

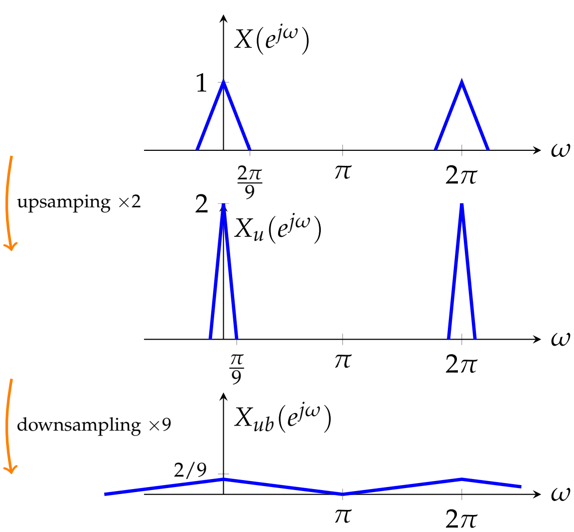
Sampling with $T/2$:



Downsampling by a Noninteger (but Rational) N

Write $N = M/L$ where M and L are integers. Upsample by L, then downsample by M.

<u>Example:</u> The signal with spectrum below can be downsampled by $N = 4.5$ without aliasing: first upsample by 2, then downsample by 9.



What happens if we downsample first, upsample next?