# *EE120 - Fall'15 - Lecture 10 Notes*[1]

*Murat Arcak*

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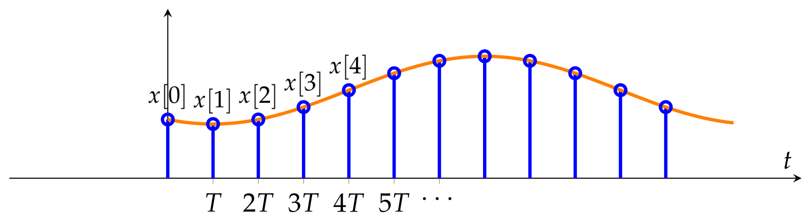
## *Sampling*

Chapter 7 *in* Oppenheim & Willsky

Discrete-time sequence obtained from continuous-time signal $x(t)$:

$$x_d[n] = x(nT), \quad T : \text{sampling period}$$



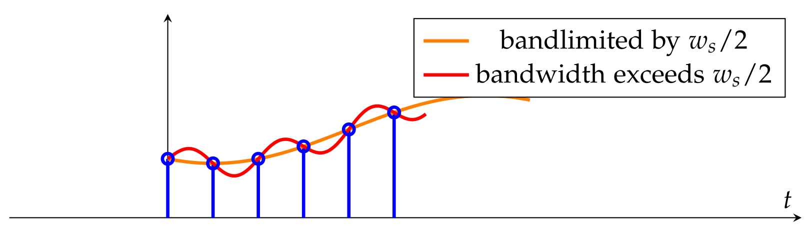

Can $x(t)$ be recovered from its samples?

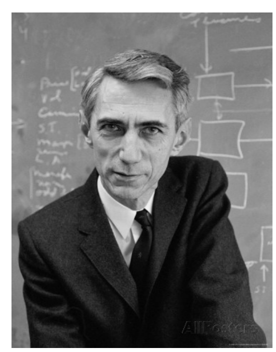
### *Shannon-Nyquist Sampling Theorem*

If $x(t)$ is bandlimited with $X(j\omega) = 0$ for $|\omega| > \omega_M$ and

$$\boxed{\omega_S > 2\omega_M} \tag{1}$$

where $w_s = \frac{2\pi}{T}$, then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \mp 1, \mp 2, \ldots$.



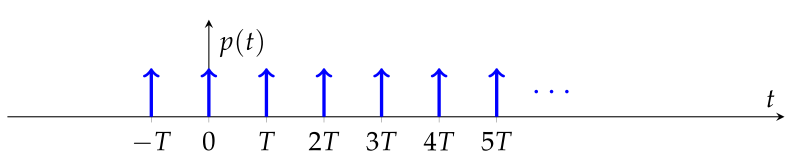
Claude Shannon (1916-2001)

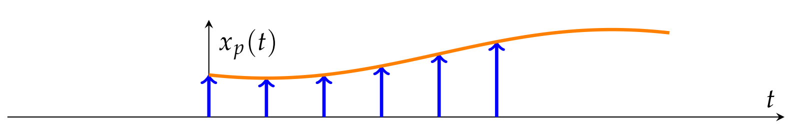
Harry Nyquist (1889-1976)

To see where (1) comes from, we define "impulse train sampling":

$$x_p(t) = x(t) \cdot p(t) \tag{2}$$

$$\text{where} \qquad p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{3}$$



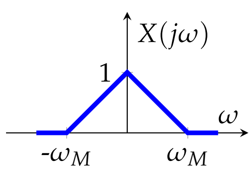
*How is $X_p(j\omega)$ related to $X(j\omega)$?*

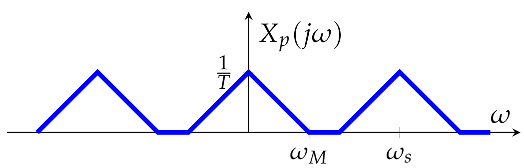
Recall: Fourier series coefficients of the impulse train are $a_k = \frac{1}{T}$ for all $k$.

$$P(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_s) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \tag{4}$$

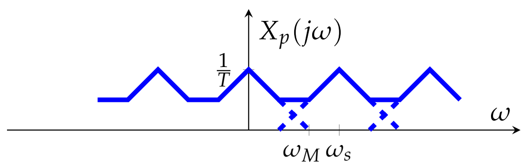
$$X_p(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\omega - \theta)) d\theta = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \tag{5}$$

$$\boxed{X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))} \tag{6}$$



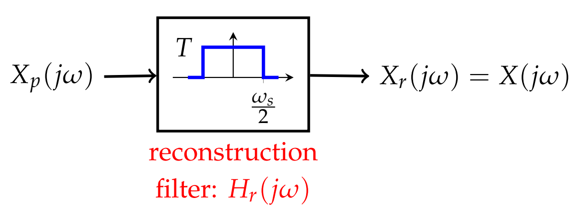




if $w_s > 2w_M$



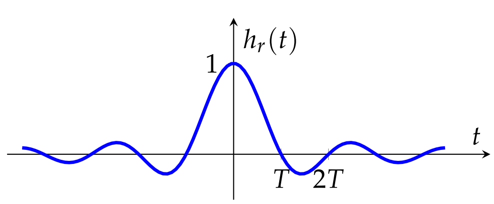

if $w_s < 2w_M$ "aliasing"

Thus, if (1) holds (no aliasing), then $x(t)$ can be recovered from $x_p(t)$ with a lowpass filter of gain $T$ and cutoff frequency $\frac{w_s}{2}$:




*The reconstruction filter is an interpolator in the time domain:*

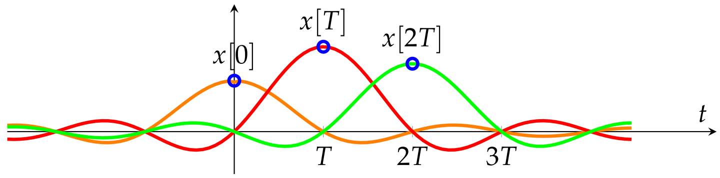
$$h_r(t) = T \frac{\sin\left(\overbrace{\frac{\omega_s}{2}}^{=\pi/T} t\right)}{\pi t} = \frac{\sin\left(\frac{\pi t}{T}\right)}{\left(\frac{\pi t}{T}\right)}$$



$$x_r(t) = h_r(t) * x_p(t) = h_r(t) * \left( \sum_n x(nT)\delta(t - nT) \right) \quad (7)$$

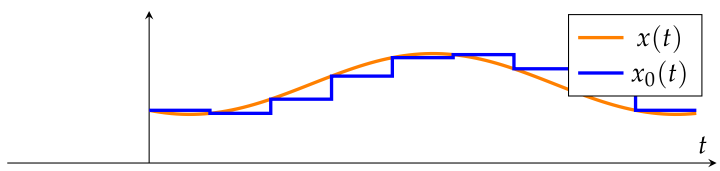
$$= \sum_n x(nT) \left( h_r(t) * \delta(t - nT) \right) \quad (8)$$

$$= \sum_n x(nT) h_r(t - nT) \quad (9)$$

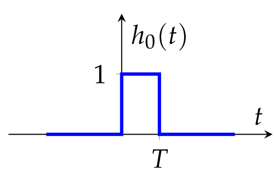



The sum of these sinc functions gives the reconstructed signal $x_r(t)$.

*Zero-order Hold Approximate Reconstruction*

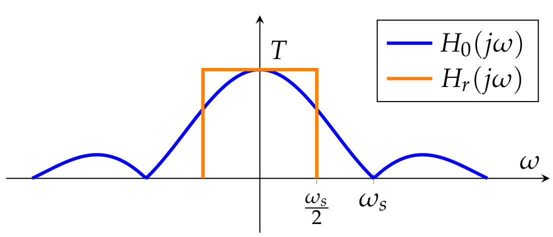



$$x_0(t) = x_p(t) * h_0(t) \quad (10)$$

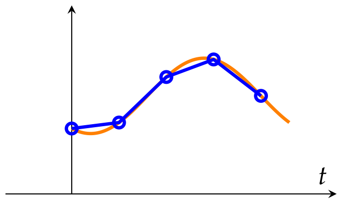



$$H_0(j\omega) = e^{-j\omega T/2} \frac{\sin(wT/2)}{w/2} \quad (11)$$

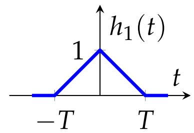
$$X_0(j\omega) = H_0(j\omega) X_p(j\omega) \quad (12)$$



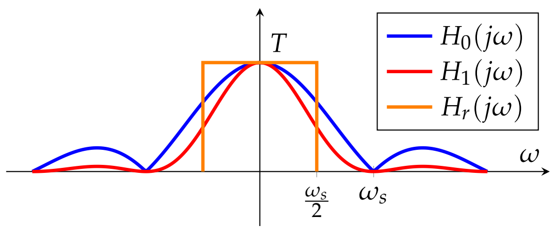
*Linear Interpolation (First-order Hold)*




$= x_p(t) * h_1(t)$



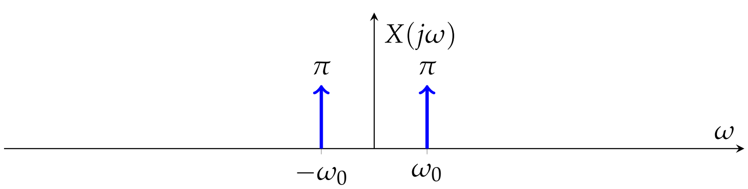

$$H_1(j\omega) = \frac{1}{T}\left(\frac{\sin(\omega T/2)}{\omega/2}\right)^2 \tag{13}$$

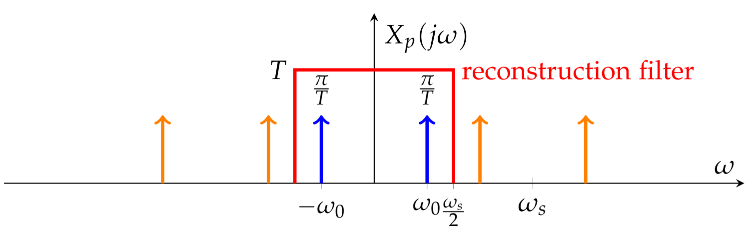


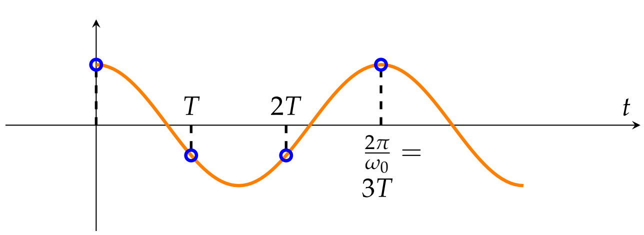

## *Examples of Aliasing*

Section 7.3 *in* Oppenheim & Willsky

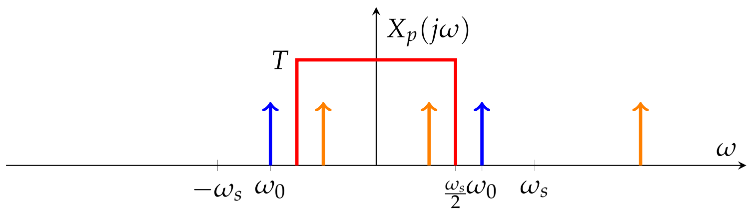
Example 1: $x(t) = \cos(\omega_0 t), \quad \omega_s = 3\omega_0$



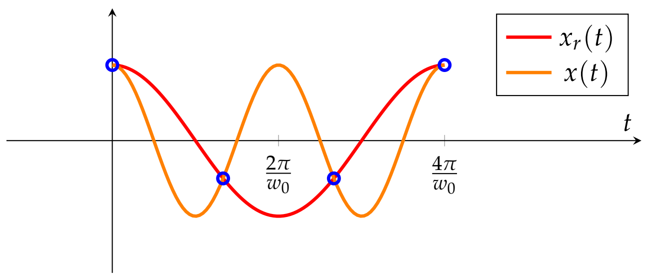



Example 2: $x(t) = \cos(\omega_0 t), \quad \omega_s = \frac{3\omega_0}{2}$

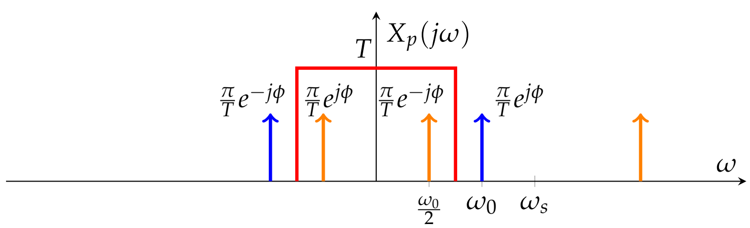



$$x_r(t) = \cos\left(\frac{\omega_0}{2} t\right) \neq x(t) \tag{14}$$




Example 3: (phase reversal)

$$x(t) = \cos(\omega_0 t + \phi) = \frac{1}{2} e^{j\phi} \underset{\substack{\updownarrow \\ 2\pi\delta(\omega - \omega_0)}}{e^{j\omega_0 t}} + \frac{1}{2} e^{-j\phi} e^{-j\omega_0 t}, \qquad \omega_s = \frac{3\omega_0}{2} \tag{15}$$




$$X_r(j\omega) = \pi e^{j\phi} \underset{\substack{\updownarrow \\ \frac{1}{2\pi} e^{-j\frac{\omega_0}{2}t}}}{\delta\left(\omega + \frac{\omega_0}{2}\right)} + \pi e^{-j\phi} \underset{\substack{\updownarrow \\ \frac{1}{2\pi} e^{j\frac{\omega_0}{2}t}}}{\delta\left(\omega - \frac{\omega_0}{2}\right)} \tag{16}$$

$$x_r(t) = \frac{1}{2}\left(e^{j\left(\frac{\omega_0}{2}t - \phi\right)} + e^{-j\left(\frac{\omega_0}{2}t - \phi\right)}\right) \tag{17}$$

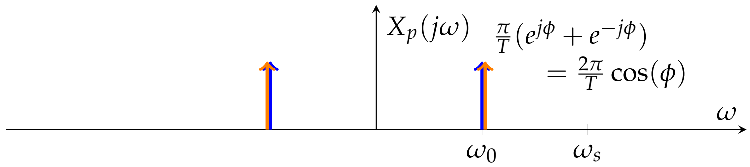
$$= \cos\left(\frac{\omega_0}{2} t - \phi\right) \rightarrow \text{phase reversal} \tag{18}$$

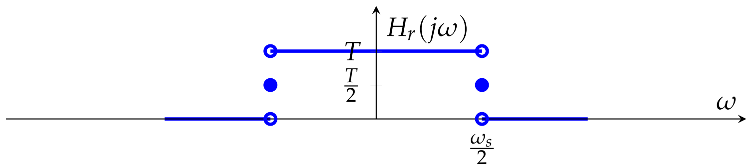
$$= \cos\left(-\frac{\omega_0}{2} t + \phi\right) \tag{19}$$

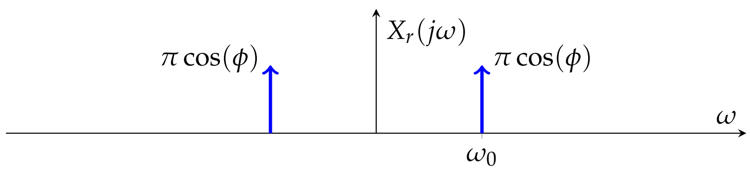
Wagon wheel effect in movies: Wheel appears to rotate more slowly and in the opposite direction when actual speed exceeds half of the sampling rate (18-24 frames/second).

Example 4: (critical frequency)

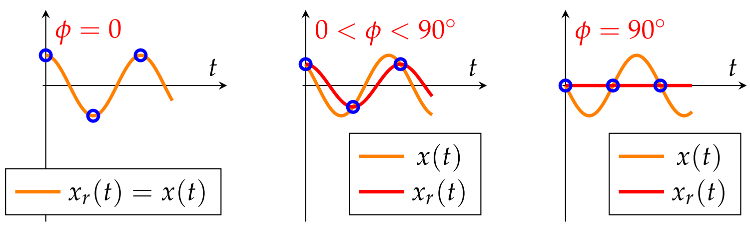
$$x(t) = \cos(\omega_0 t + \phi) \qquad \omega_s = 2\omega_0 \tag{20}$$




$$x_r(t) = \cos(\phi)\cos(\omega_0 t) \neq x(t) \quad \text{unless } \phi = 0. \tag{21}$$




Example 5: $x(t) = \cos(\omega_0 t)$ $w_s = 3\omega_0$, zero-order hold reconstruction