EE120 - Fall'15 - Lecture 1 Notes[1]

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Linear Time-Invariant (LTI) Systems

Linearity: Two conditions must be satisfied:

1. Scaling:

$$ax(t) \to ay(t) \quad \text{for any number} \quad a; \tag{1}$$

2. Superposition:

$$x_1(t) + x_2(t) \to y_1(t) + y_2(t). \tag{2}$$

Corollary: If the input to a linear system is 0, the output must be 0.
Proof. Choose $a = 0$ in the scaling property.

Time-Invariance:A time shift in the input results is an identical time shift in the output:

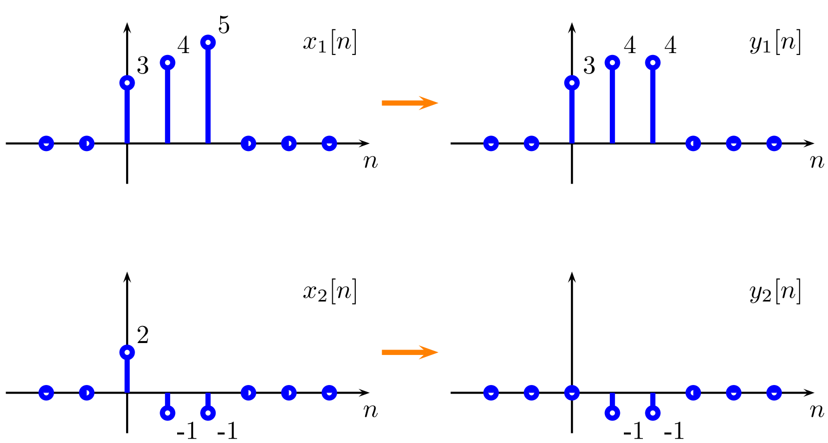
$$x(t-T) \to y(t-T). \tag{3}$$

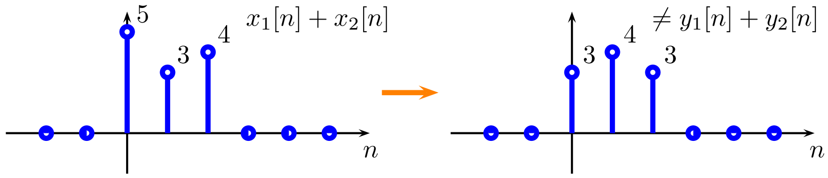
Example: Moving average filter:

$$y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right) \to \text{ LTI.} \tag{4}$$

Example: Median Filter:

$$y[n] = med\{x[n-1], x[n], x[n+1]\} \to \text{ TI, but nonlinear.} \tag{5}$$

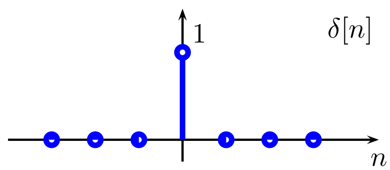




Discrete-Time (DT) LTI Systems: Convolution Sum

Section 2.1 *in* Oppenheim & Willsky

Let $h[n]$ denote the response of an LTI system to the unit impulse:



Then, for any input $x[n]$, the output is:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad \text{"convolution sum"} \tag{6}$$

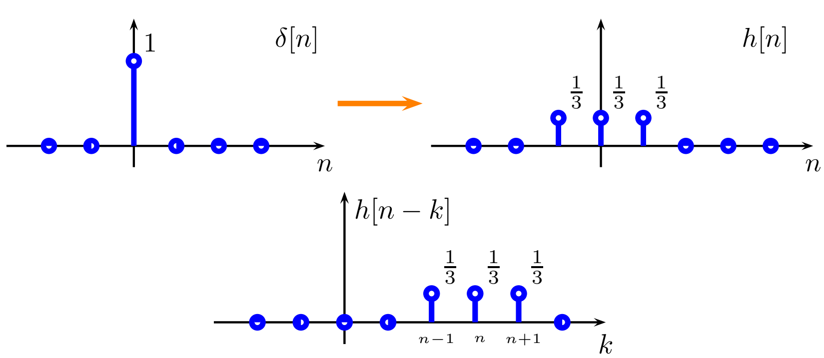
Proof. Rewrite $x[n]$ as

$$x[n] = ... + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + ... \tag{7}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{8}$$

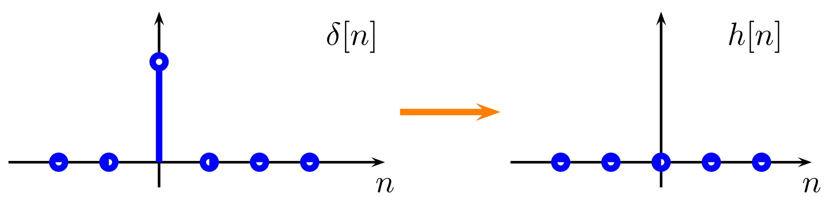
Since $\delta[n] \to h[n]$, by time-invariance: $\delta[n-k] \to h[n-k]$.
Then, by linearity: $\sum_k x[k]\delta[n-k] \to \sum_k x[k]h[n-k]$.

Example: For the moving average system above,



$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=n-1}^{n+1} \frac{1}{3}x[k] = \frac{1}{3}\left(x[n-1]+x[n]+x[n+1]\right) \end{aligned} \tag{9}$$

Example: For the median filter:



Since the system is nonlinear, we can't use convolution to predict the output.

Continuous-Time (CT) LTI Systems: Convolution Integral

Section 2.2 *in* Oppenheim & Willsky

Unit impulse:

$$\delta(t) \triangleq \lim_{\Delta\to 0} \delta_\Delta(t) \tag{10}$$

where $\delta_\Delta(t)$ is as in Figure 1.

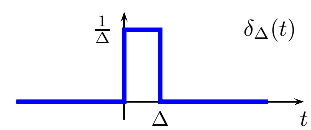


Figure 1: $\delta_\Delta(t)$

Let $h(t)$ denote the response of a LTI system to $\delta(t)$.
Then, for any input $x(t)$, the output is :

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad \text{"convolution integral"} \tag{11}$$

Proof. First, note that the staircase approximation in Figure 2 recovers $x(t)$ as $\Delta \to 0$:

$$x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta\delta_\Delta(t-k\Delta). \tag{12}$$

Next, let $h_\Delta(t)$ denote the response of the system to $\delta_\Delta(t)$ and note from the LTI property that the response to each term in the sum above is $x(k\Delta)\Delta h_\Delta(t-k\Delta)$. Thus, the response to $x(t)$ is

$$y(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t-k\Delta)\Delta = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau. \tag{13}$$

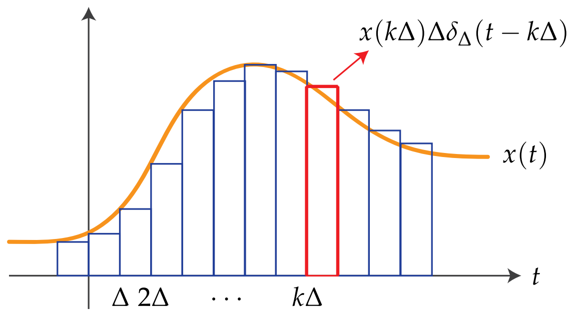


Figure 2: Staircase approximation of $x(t)$.

Properties of LTI Systems

Section 2.3 *in* Oppenheim & Willsky

We will denote the convolution operation by $''*''$.

1. Commutative Property:

$$\boxed{x[n] * h[n] = h[n] * x[n]} \tag{14}$$

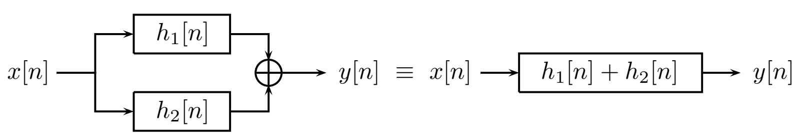
Proof.

$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{r=-\infty}^{\infty} x[n-r]h[r], \tag{15}$$

with the change of variables $(n-k) \triangleq r$.

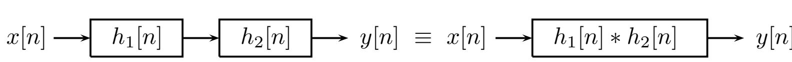
2. Distributive Property:

$$\boxed{x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]} \tag{16}$$



3. Associative Property:

$$\boxed{x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]} \tag{17}$$



Combine this with the commutative property

Properties 1,2,3 above also hold for CT systems.

Determining Causality from the Impulse Response

For a DT LTI system, causality means:

$$\boxed{h[n] = 0, \quad \forall n < 0.} \tag{18}$$

For a CT LTI system, causality means:

$$\boxed{h(t) = 0, \quad \forall t < 0.} \tag{19}$$

Proof. Since $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$, if $h[k] \neq 0$ for some $k < 0$, then $y[n]$ depends on $x[n-k]$, where $n - k > n$.

Example: Moving average system above: $h[-1] \neq 0 \rightarrow$ noncausal.

Determining Stability from the Impulse Response

Stability criterion for a DT LTI system:

$$\boxed{\sum_{k=-\infty}^{\infty} |h[k]| < \infty.} \tag{20}$$

Stability criterion for a CT LTI system:

$$\boxed{\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty.} \tag{21}$$

Proof.

Sufficiency: Suppose $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ and show that bounded inputs give bounded outputs:

$|x[n]| \leq B$ for all n, for some $B > 0$.

$|y[n]| = |\sum_k x[n-k]h[k]| \leq \sum_k |x[n-k]| \cdot |h[k]| \leq B \sum_k |h[k]| < \infty.$

Necessity: To prove "stable $\Rightarrow \sum_k |h[k]| < \infty$" prove the contrapositive:

$$\text{"} \sum_k |h[k]| = \infty \Rightarrow \text{ unstable."} \tag{22}$$

Let $x[n] = sgn\{h[-n]\}$. Then, since $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$:

$$y[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_k h[k]sign\{h[k]\} = \sum_k |h[k]| = \infty. \tag{23}$$

Examples:

1. Moving average system above:

$$\sum_k |h[k]| = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1 \rightarrow \text{ stable.} \tag{24}$$

2. Integrator: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$. $h(t)$ is the unit step (see Figure 3), and

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \infty. \tag{25}$$

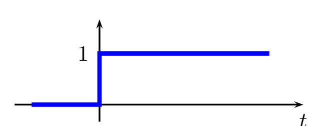


Figure 3: *UnitStep*