1 a) $x(0^+) = \lim_{s\to\infty} \frac{s(s+1)}{(s+2)(s+3)} = 1$

$\lim_{t\to\infty} x(t) = \lim_{s\to 0} \frac{s(s+1)}{(s+2)(s+3)} = 0$

b) $x(0^+) = \lim_{s\to\infty} \frac{s}{s(s+1)} = 0$

$\lim_{t\to\infty} x(t) = \lim_{s\to 0} \frac{s}{s(s+1)} = 1$

c) $\frac{s-1}{s+1} = 1 - \frac{2}{s+1} \xleftrightarrow{\mathcal{L}^{-1}} \delta(t) - 2e^{-t}u(t)$

$x(0^+) = -2$

$\lim_{t\to\infty} x(t) = 0$

d) $x(0^+) = \lim_{s\to\infty} \frac{s}{(s+2)(s+3)} = 0$

$\lim_{t\to\infty} x(t) = \lim_{s\to 0} \frac{s}{(s+2)(s+3)} = 0$

3 a) $y[n] = h[n] * (u[n] - u[n-N-1])$

$= h[n]*u[n] - h[n]*u[n-N-1]$

$= \underbrace{h[n]*u[n]}_{g[n]} - \underbrace{h[n]*u[n]*\delta[n-N-1]}_{g[n-N-1]}$

$g[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k]$

$= \sum_{k=0}^{\infty} \alpha^k u[n-k]$

$= u[n] \cdot \sum_{k=0}^{n} \alpha^k$

$= u[n] \cdot \frac{1-\alpha^{n+1}}{1-\alpha}$

$y[n] = u[n] \cdot \frac{1-\alpha^{n+1}}{1-\alpha} - u[n-N-1] \cdot \frac{1-\alpha^{n-N}}{1-\alpha}$

b) $H(z) = \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$

$X(z) = \frac{1-z^{-N-1}}{1-z^{-1}}, \quad |z| > 1$

$Y(z) = \frac{1-z^{-N-1}}{(1-z^{-1})(1-\alpha z^{-1})}$

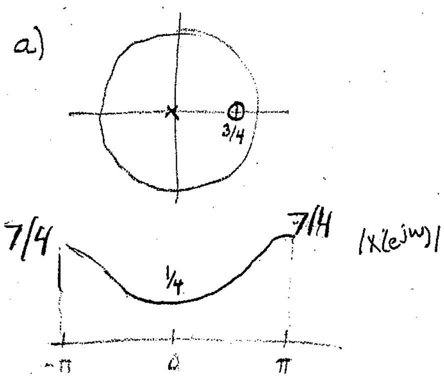
$= \frac{A}{1-z^{-1}} + \frac{B}{1-\alpha z^{-1}} - z^{-N-1}\left[\frac{A}{1-z^{-1}} + \frac{B}{1-\alpha z^{-1}}\right]$

where $A = \frac{1}{1-\alpha}, \quad B = -\frac{\alpha}{1-\alpha}$

$y[n] = Au[n] + B\alpha^n u[n] - Au[n-N-1] - B\alpha^{n-N-1}u[n-N-1]$

$=$

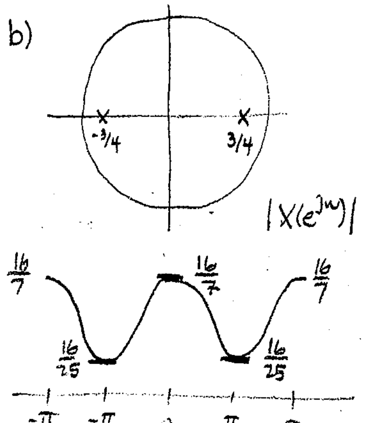
5 a)



$|X(e^{j0})| = \frac{|1-3/4|}{1} = \frac{1}{4}$

$|X(e^{j\pi})| = \frac{|1+3/4|}{1} = \frac{7}{4}$

b)

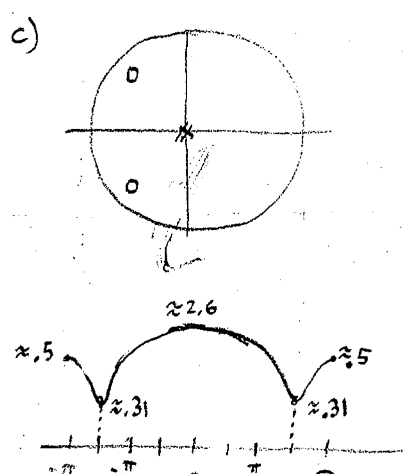


$|X(e^{j0})| = \frac{1}{\frac{1}{4}\cdot\frac{7}{4}} = \frac{16}{7}$

$|X(e^{j\pi/2})| = \frac{1}{|j+3/4|^2}$

$= \frac{1}{\frac{25}{16}} = \frac{16}{25}$

c)



numerically

(2)

$$y[n] - 0.5y[n-1] + 0.25y[n-2] = x[n].$$

$$y[n] \xleftrightarrow{Z.T.} Y(z)$$

$$y[n-1] \longleftrightarrow z^{-1}Y(z) + \underbrace{y[-1]}_{=3}$$

$$y[n-2] \longleftrightarrow z^{-2}Y(z) + z^{-1}\underbrace{y[-1]}_{=3} + \underbrace{y[-2]}_{=1}.$$

$$Y(z) - \frac{1}{2}\left[z^{-1}Y(z) + 3\right] + \frac{1}{4}\left[z^{-2}Y(z) + 3z^{-1} + 1\right] = \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{X(z)}$$

$$Y(z)\left[1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right] + \frac{3}{4}z^{-1} + \overbrace{\frac{1}{4} - \frac{3}{2}}^{-5/4} = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$Y(z) = \frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}\left[\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{5}{4} - \frac{3}{4}z^{-1}\right]$$

$$Y(z) = \frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}\left[\frac{1+\frac{5}{4}-\frac{5}{8}z^{-1}-\frac{3}{4}z^{-1}+\frac{3}{8}z^{-2}}{1-\frac{1}{2}z^{-1}}\right]$$

$$Y(z) = \frac{\frac{3}{8}z^{-2} - \frac{11}{8}z^{-1} + \frac{9}{4}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right)}$$

PFE: $$Y(z) = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{Bz^{-1}+C}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$$

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$$A\left(1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}\right) + \left(Bz^{-1}+C\right)\left(1-\frac{1}{2}z^{-1}\right) = \frac{3}{8}z^{-2} - \frac{11}{8}z^{-1} + 9/4$$

$z^{-1}=2$: $A(1-1+1) = \frac{3}{2} - \frac{11}{4} + \frac{9}{4} \Rightarrow A = 1$

$z^{-1}=1$: $\left(1-\frac{1}{2}+\frac{1}{4}\right) + (B+C)\left(\frac{1}{2}\right) = \frac{3}{8} - \frac{11}{8} + 9/4$

$$\left(\frac{1}{2}\right)(B+C) = \frac{5}{4} - \frac{3}{4} \Rightarrow B+C = 1 \text{ — (1)}$$

$z^{-1}=-1$: $\left(1+\frac{1}{2}+\frac{1}{4}\right) + (-B+C)\left(\frac{3}{2}\right) = \frac{3}{8} + \frac{11}{8} + \frac{9}{4}$

$$\left(\frac{3}{2}\right)(-B+C) = 4 - \frac{7}{4} = \frac{9}{4}$$

$$-B+C = 3/2 \text{ — (2)}$$

From (1) & (2): $2C = 5/2 \rightarrow C = 5/4$

$\therefore B = -1/4$

$$\therefore Y(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{\frac{5}{4} - \frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$$

→ consult z-Transform table

$2r\cos(\omega_0)$; $r^2 = 1/4$; $r = 1/2$

$\hookrightarrow \omega_0 = \pi/3 \rightarrow \sin(\omega_0)$

$$\frac{5}{4}\,\frac{1-\frac{1}{4}z^{-1}+\frac{1}{20}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}} = \frac{5}{4}\,\frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}} + \frac{5}{80}\,\frac{\frac{4}{\sqrt{3}}\cdot z^{-1}\cdot\frac{\sqrt{3}}{4}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$$

* manipulated this to get it in the form available in the table.

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$$\therefore Y(z) = \left[\frac{1}{1-\frac{1}{2}z^{-1}}\right] + \left[\frac{5}{4}\,\frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}\right] + \frac{1}{4\sqrt{3}}\cdot\left[\frac{\frac{\sqrt{3}}{4}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}\right]$$

Hence, from table,

$$y[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{5}{4}\left(\frac{1}{2}\right)^n \cos\left(\frac{\pi}{3}n\right)u[n]$$
$$+ \frac{1}{4\sqrt{3}}\left(\frac{1}{2}\right)^n \sin\left(\frac{\pi}{3}n\right)u[n]$$

$$\boxed{y[n] = \left(\frac{1}{2}\right)^n\left(1+\frac{5}{4}\cos\left(\frac{\pi}{3}n\right)+\frac{1}{4\sqrt{3}}\sin\left(\frac{\pi}{3}n\right)\right)u[n]}$$

④

a) $H(z) = \dfrac{z^{-1}-\frac{1}{2}}{1-\frac{1}{2}z^{-1}}$

i)

$$\begin{array}{r|l} & -\frac{1}{2}+\frac{3}{4}z^{-1}+\frac{3}{8}z^{-2}+\dots \\ \hline 1-\frac{1}{2}z^{-1} & -\frac{1}{2}+z^{-1} \\ & -\frac{1}{2}+\frac{1}{4}z^{-1} \\ & +\quad - \\ \hline & \frac{3}{4}z^{-1} \\ & \frac{3}{4}z^{-1}-\frac{3}{8}z^{-2} \\ & -\quad + \\ \hline & \frac{3}{8}z^{-2} \\ & \frac{3}{8}z^{-2}-\frac{3}{16}z^{-3} \\ & -\quad + \\ \hline & \vdots \end{array}$$

$$\therefore H(z) = -\frac{1}{2}+\frac{3}{4}z^{-1}+\frac{3}{8}z^{-2}+\frac{3}{16}z^{-3}+\dots$$
$$\updownarrow$$
$$h[n] = -\frac{1}{2}\delta[n]+\frac{3}{4}\delta[n-1]+\frac{3}{8}\delta[n-2]+\frac{3}{16}\delta[n-3]+\dots$$
$$h[n] = -\frac{1}{2}\delta[n]+\frac{3}{2^{n+1}}u[n-1]$$

ii) $H(z) = -\dfrac{\frac{1}{2}}{1-\frac{1}{2}z^{-1}} + \dfrac{z^{-1}}{1-\frac{1}{2}z^{-1}}$

$$\therefore h[n] = -\frac{1}{2}\left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)^{n-1}u[n-1]$$
$$h[n] = -\frac{1}{2}\delta[n] + u[n-1]\underbrace{\left(\left(2-\frac{1}{2}\right)\left(\frac{1}{2}\right)^n\right)}_{3\cdot\frac{1}{2^{n+1}}}$$

b) $H(z) = \dfrac{z^{-1} - 1/2}{\left(1 - \frac{1}{2}z^{-1}\right)^2} = \dfrac{z^{-1} - 1/2}{1 - z^{-1} + \frac{1}{4}z^{-2}}$

i)

$$\begin{array}{r|l}
 & -\frac{1}{2} + \frac{1}{2}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{2}z^{-3} + \frac{11}{32}z^{-4} + \dots \leftarrow H(z) \\
1 - z^{-1} + \frac{1}{4}z^{-2} & -\frac{1}{2} + z^{-1} \\
 & -\frac{1}{2} + \frac{1}{2}z^{-1} - \frac{1}{8}z^{-2} \\
 & \quad + \quad - \quad + \\ \hline
 & \frac{1}{2}z^{-1} + \frac{1}{8}z^{-2} \\
 & \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2} + \frac{1}{8}z^{-3} \\
 & \quad - \quad + \quad - \\ \hline
 & \frac{5}{8}z^{-2} - \frac{1}{8}z^{-3} \\
 & \frac{5}{8}z^{-2} - \frac{5}{8}z^{-3} + \frac{5}{32}z^{-4} \\
 & \quad - \quad + \quad - \\ \hline
 & \frac{1}{2}z^{-3} - \frac{5}{32}z^{-4} \\
 & \frac{1}{2}z^{-3} - \frac{1}{2}z^{-4} + \frac{1}{8}z^{-5} \\
 & \quad - \quad + \quad - \\ \hline
 & \frac{11}{32}z^{-4} - \frac{1}{8}z^{-5}
\end{array}$$

$\therefore h[n] = -\frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1] + \frac{5}{8}\delta[n-2] + \frac{1}{2}\delta[n-3] + \frac{11}{32}\delta[n-4] + \dots$

ii) $H(z) = \dfrac{A}{1 - \frac{1}{2}z^{-1}} + \dfrac{B}{\left(1 - \frac{1}{2}z^{-1}\right)^2}$

$A\left(1 - \frac{1}{2}z^{-1}\right) + B = -1/2 + z^{-1}$

$z^{-1} = 2$: $B = 3/2$

$z^{-1} = 1$: $\frac{A}{2} + \frac{3}{2} = \frac{1}{2} \rightarrow \frac{A}{2} = -1 \Rightarrow A = -2$

$H(z) = \dfrac{-2}{1 - \frac{1}{2}z^{-1}} + \dfrac{3/2}{\left(1 - \frac{1}{2}z^{-1}\right)^2}$

$n a^n u[n] \xleftrightarrow{Z.T.} \dfrac{a z^{-1}}{(1 - a z^{-1})^2}$

$\Downarrow$

$(n+1) a^{n+1} u[n+1] \xleftrightarrow{Z.T.} \dfrac{a}{(1 - a z^{-1})^2}$

$\therefore h[n] = -2\left(\frac{1}{2}\right)^n u[n] + \frac{3}{2} \times 2(n+1)\left(\frac{1}{2}\right)^{n+1} u[n+1]$

$h[-1] = 0$

$\therefore h[n] = u[n]\left(\frac{3}{2}(n+1)\left(\frac{1}{2}\right)^n - 2\cdot\left(\frac{1}{2}\right)^n\right)$

$h[n] = u[n]\left(\frac{3}{2}n - \frac{1}{2}\right)\left(\frac{1}{2}\right)^n$

$h[n] = \{\underset{\underset{n=0}{\uparrow}}{-\frac{1}{2}}, \frac{1}{2}, \frac{5}{8}, \frac{1}{2}, \frac{11}{32}, \dots\}$

c) $H(z) = \dfrac{1-z^{-1}}{1-\frac{1}{4}z^{-2}} = \dfrac{1-z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}$

i)

$$
\begin{array}{r|l}
 & 1 - z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{4}z^{-3} + \frac{1}{16}z^{-4} + \cdots \\
z^2 - 1/4 & z^2 - z \\
 & z^2 + 0 - 1/4 \\
\hline
 & -z + 1/4 \\
 & -z + 0 + \frac{1}{4}z^{-1} \\
\hline
 & \frac{1}{4} - \frac{1}{4}z^{-1} \\
 & \frac{1}{4} + 0 - \frac{1}{16}z^{-2} \\
\hline
 & -\frac{1}{4}z^{-1} + \frac{1}{16}z^{-2} \\
 & -\frac{1}{4}z^{-1} + 0 + \frac{1}{16}z^{-3} \\
\hline
\end{array}
$$

$\therefore H(z) = 1 - z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{4}z^{-3} + \frac{1}{16}z^{-4} + \cdots$

$h[n] = \delta[n] - \delta[n-1] + \frac{1}{4}\left(\delta[n-2] - \delta[n-3]\right) + \cdots$

ii) $H(z) = \dfrac{A^{=3/2}}{1+\frac{1}{2}z^{-1}} + \dfrac{B^{=-1/2}}{1-\frac{1}{2}z^{-1}}$

$A\left(1-\frac{1}{2}z^{-1}\right) + B\left(1+\frac{1}{2}z^{-1}\right) = 1 - z^{-1}$

$z^{-1} = 2$: $2B = -1 \rightarrow B = -1/2$

$z^{-1} = -2$: $2A = 3 \rightarrow A = 3/2$.

$\therefore h[n] = \frac{3}{2}\left(-\frac{1}{2}\right)^n u[n] - \frac{1}{2}\left(\frac{1}{2}\right)^n u[n]$

matches result for $h[n]$ above.

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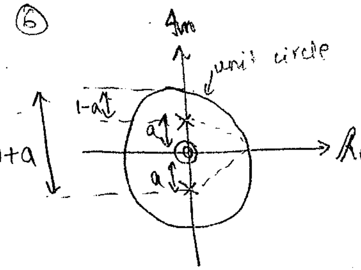
(6)



$H(e^{j0}) = \dfrac{K}{1+a^2}$

$H(e^{j\pi/2}) = \dfrac{K}{(1-a)(1+a)}$

$\dfrac{H(e^{j\pi/2})}{H(e^{j0})} = 4 = \dfrac{1+a^2}{1-a^2} \Rightarrow 1+a^2 = 4-4a^2$

$\Rightarrow 5a^2 = 3$

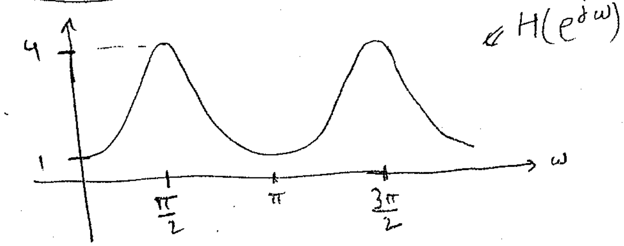
$a = \sqrt{3/5}$

$H(e^{j0}) = \dfrac{K}{1+\frac{3}{5}} = 1 \Rightarrow K = 8/5$.

$\therefore H(z) = \dfrac{\frac{8}{5}z^2}{(z+ja)(z-ja)} = \dfrac{8/5}{\left(1+j\sqrt{\frac{3}{5}}z^{-1}\right)\left(1-j\sqrt{\frac{3}{5}}z^{-1}\right)}$

$Y(z)\left(1+\frac{3}{5}z^{-2}\right) = \frac{8}{5}X(z)$

$\therefore \boxed{y[n] + \frac{3}{5}y[n-2] = \frac{8}{5}x[n]}$



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