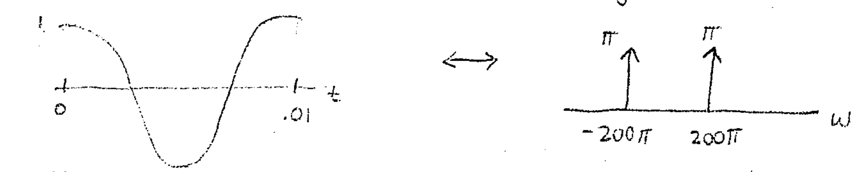
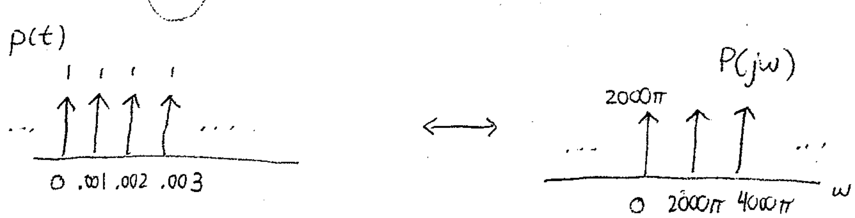
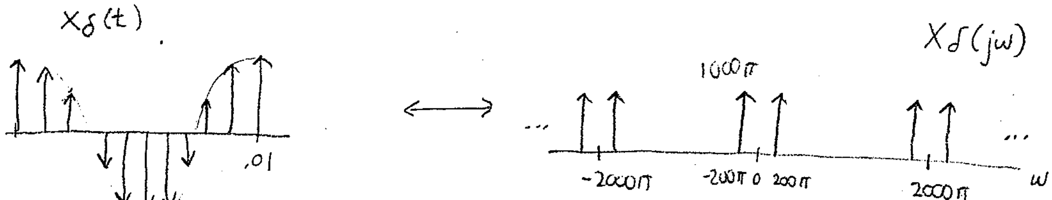
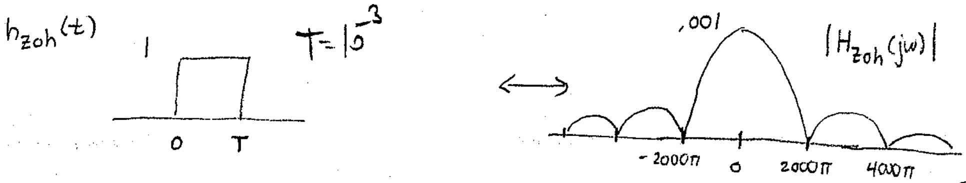
EE120 Fall 2014 PS #7

(1) a) $x(t) = \cos(2000\pi t)$

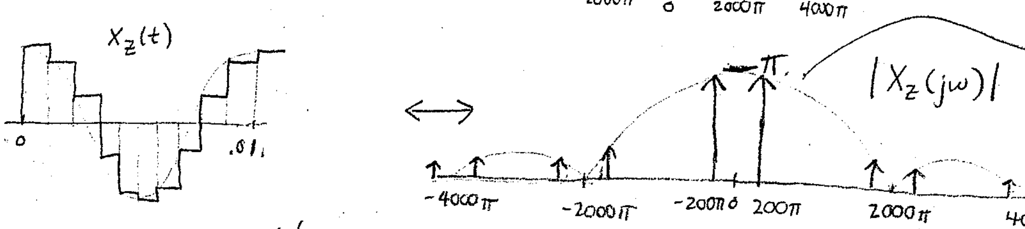






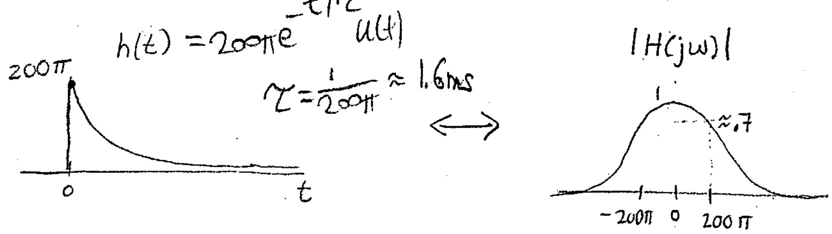


$$H_{zoh}(j\omega) = e^{-j\omega/2000}\frac{\sin \omega/2000}{\omega/2}$$

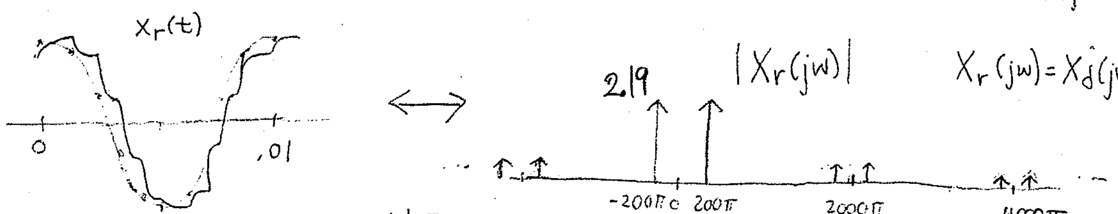


$$2000\pi \cdot \frac{\sin \pi/10}{200\pi}$$

≈ 3.1



$$H(j\omega) = \int h(t) e^{-j\omega t} dt$$
$$= \int_0^\infty 200\pi e^{-200\pi t} e^{-j\omega t} dt$$
$$= \frac{200\pi}{200\pi + j\omega}$$



$$X_r(j\omega) = X_\delta(j\omega) H_{zoh}(j\omega) H(j\omega)$$

$\omega_0 = 200\pi$

b) Note that $x_r(t)$ is periodic, so an impulse in $X_r(j\omega)$ at $\omega_k = 200\pi \cdot k$ with magn. B_k corresponds to Fourier series coefficient a_k with magn $\frac{B_k}{2\pi}$.

$$P_{fund} = |a_1|^2 + |a_{-1}|^2 = 2|a_1|^2 = 2 \cdot \left(\frac{B_1}{2\pi}\right)^2$$

$\approx .241883$

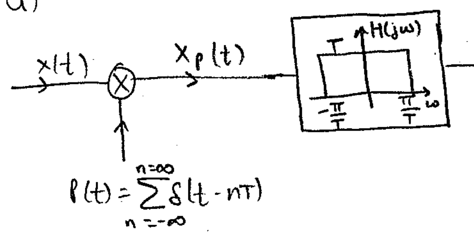
$$a_9 = \frac{1}{2\pi} X_r(j9 \cdot 200\pi) = \frac{1}{2\pi} X_z(j1800\pi) \cdot H(j1800\pi)$$
$$= \frac{1}{2\pi}(0.34)(0.11)$$
$$= 6 \times 10^{-3}$$

$$P_{all} = \sum_k |a_k|^2 = \sum_k \left(\frac{B_k}{2\pi}\right)^2 = \sum_{\substack{k = \pm 1, \pm 9, \pm 11, \\ \pm 19, \pm 21, \ldots}} \frac{1}{4\pi^2} \cdot \left[1000\pi \cdot \frac{\sin \omega_k/2000}{\omega_k/2} \cdot \frac{200\pi}{\sqrt{(200\pi)^2 + \omega^2}}\right]^2 \approx .241996$$

Fraction of power not at ± 100 Hz $\approx 0.05\% = \frac{P_{all} - P_{fund}}{P_{all}}$

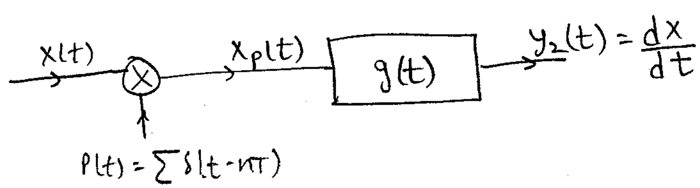
②

a)



Accurately recovered with ideal LPF used as interpolator

This diagram illustrates the problem we've looked at thus far in the course. The following diagram represents the scenario under consideration here.



$$x_p(t) = \sum_{n=-\infty}^{n=\infty} x(nT)\,\delta(t-nT).$$

$$y_2(t) = g(t) * \sum_{n=-\infty}^{n=\infty} x(nT)\,\delta(t-nT)$$

$$y_2(t) = \sum_{n=-\infty}^{n=\infty} x(nT)\,g(t-nT)$$

note that in the first diagram,

$$y_1(t) = \sum_{n=-\infty}^{n=\infty} x(nT)\,h(t-nT)$$

So, to obtain $y_2(t) = \frac{dx}{dt}$, we use

$g(t) = \frac{dh}{dt}$

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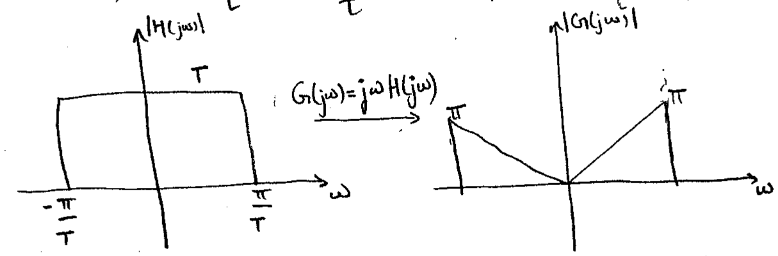
The ideal LPF interpolator in first diagram is $h(t) = \frac{T}{\pi} \frac{\sin(\pi t/T)}{t}$

So,

$$g(t) = \frac{T}{\pi} \frac{d}{dt}\left[\sin\left(\frac{\pi t}{T}\right) t^{-1}\right]$$

$$g(t) = \frac{T}{\pi}\left[\frac{-\sin(\pi t/T)}{t^2} + \frac{\pi}{T}\frac{\cos(\pi t/T)}{t}\right]$$



b) → Based on the above, $g(t) = \frac{dh}{dt}$ ensures that $y_2(t) = \frac{dy_1}{dt}$, referring to diagrams shown earlier.

→ For ideal reconstruction, i.e. $y_1(t) = x(t)$, $h(t)$ is unique thus $g(t)$ is also unique.

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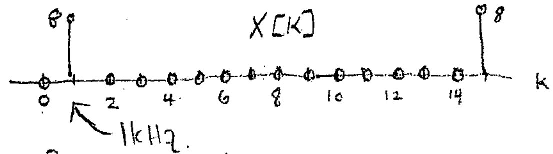
3 a) $x(t) = \cos(2\pi \cdot 1000\, t)$

$x[n] = x(nT) = \cos\left(2\pi \cdot 1000\, n \cdot \frac{1}{16000}\right) = \cos\left(\frac{\pi}{8} n\right) = \frac{1}{2} e^{j\frac{\pi}{8}n} + \frac{1}{2} e^{-j\frac{\pi}{8}n}$, $n = 0, 1, \dots, 15$

Recall that $x[n] = \frac{1}{16} \sum_{k=0}^{15} X[k]\, e^{j\frac{\pi}{8}kn}$

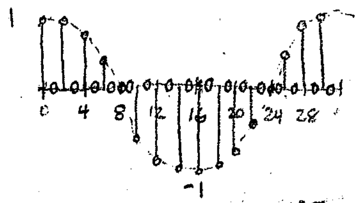
By inspection, we can see that $X[K] = \begin{cases} 8, & K = 1, 15 \\ 0 & \text{otherwise} \end{cases}$



spacing of samples: $\frac{2\pi}{N} = \frac{\pi}{8} \Rightarrow \frac{2\pi}{T_0} = 1\text{kHz}.$

$K=4$ corresponds to $\frac{2\pi k}{N} = \frac{\pi}{2} = 4 \cdot \frac{2\pi}{T_0} = 4\text{kHz}$

b)



$$Y[K] = \sum_{n=0}^{31} y[n]\, e^{-j\frac{2\pi}{32}kn}$$

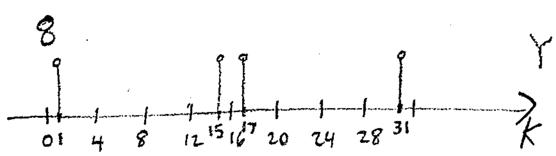
$$= \sum_{\substack{n=0 \\ n\text{ even}}}^{30} y[n]\, e^{-j\frac{\pi}{16}kn}$$

$$= \sum_{\substack{n=0, \\ n\text{ even}}}^{30} x[n/2]\, e^{-j\frac{\pi}{16}kn} \qquad m = n/2$$

$$= \sum_{m=0}^{15} x[m]\, e^{-j\frac{\pi}{8}km}$$

$$= X[K], \quad k = 0, 1, \dots, 31$$

$\leftarrow$ Note that for $k = 16, \dots, 31$, $X[K] = X[K-16]$.



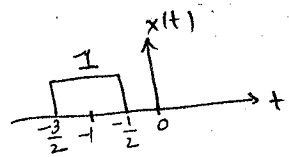
$Y[K] = \begin{cases} 8, & K = 1, 15, 17, 31 \\ 0 & \text{otherwise} \end{cases}$

Spacing of samples: $\frac{2\pi}{N} = \frac{\pi}{16}$, $k=32$ corresponds to $\frac{2\pi}{T_s'}$, where $T_s' = T_s/2 = \frac{1}{32\text{ kHz}}$. $\frac{2\pi}{T_s'} = 2\pi \cdot 32\text{kHz}$, $k=1$ corresponds to $\frac{2\pi}{\frac{32\text{kHz}}{32}} = 1\text{kHz}$.

Spacing of samples same in frequency domain as for $X[k]$.

(4)

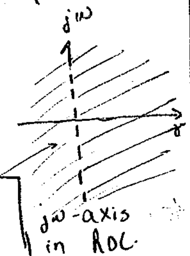
a) $x(t) = \Pi(t+1)$



$$X(s) = \int_{-3/2}^{-1/2} e^{-st}\,dt = -\frac{1}{s}\left[e^{s/2} - e^{3s/2}\right]$$

↑ converges for any s since finite limits.

$$\boxed{X(s) = -\frac{e^{s/2}}{s}\left[1 - e^{s}\right] ; \ Re\{s\} > \infty}$$



b) $x(t) = e^{-|t|} = \underbrace{e^{t}u(-t)}_{Re\{s\}<1} + \underbrace{e^{-t}u(t)}_{Re\{s\}>-1}$. $u(0) = \frac{1}{2}$, $u(0) = \frac{1}{2}$

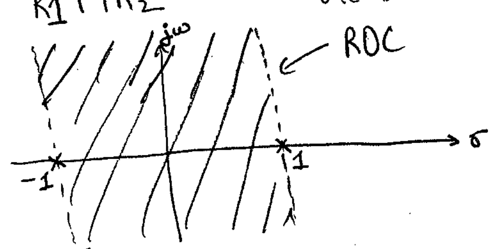
Using table,

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} ; \ Re\{s\} > -a \quad R_1$$

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} ; \ Re\{s\} < -\alpha \quad R_2$$

$$\therefore X(s) = \frac{-1}{s-1} + \frac{1}{s+1} = \frac{-2}{s^2-1}$$

ROC: $R_1 \cap R_2 \Rightarrow -1 < Re\{s\} < 1$



jw - axis is in the ROC.

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c) $x(t) = e^{2t}\cos(4\pi t)u(t)$

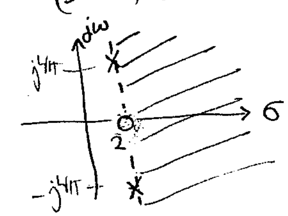
$$x(t) = e^{2t}\left[\frac{1}{2}e^{j4\pi t} + \frac{1}{2}e^{-j4\pi t}\right]u(t)$$

$$x(t) = \left[\frac{1}{2}e^{t(2+j4\pi)} + \frac{1}{2}e^{t(2-4j\pi)}\right]u(t).$$

$$X(s) = \frac{1/2}{s-2-j4\pi} + \frac{1/2}{s-2+j4\pi}$$

$$= \frac{1}{2}\left[\frac{s-2+\cancel{j4\pi} + s-2-\cancel{j4\pi}}{(s-2)^2 + 16\pi^2}\right]$$

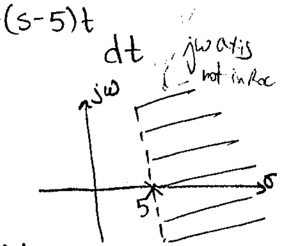
$$X(s) = \frac{s-2}{(s-2)^2+16\pi^2} ; \ Re\{s\} > 2$$



jw-axis not in the ROC.

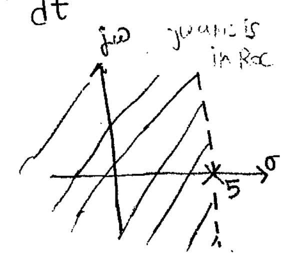
d) $x(t) = e^{5t}u(t)$; $X(s) = \int_0^\infty e^{-(s-5)t}\,dt$

$$\boxed{X(s) = \frac{1}{s-5} ; \ Re\{s\} > 5}$$



e) $x(t) = e^{5t}u(-t)$; $X(s) = \int_{-\infty}^{0} e^{-(s-5)t}\,dt$

$$\boxed{X(s) = \frac{-1}{s-5} ; \ Re\{s\} < 5}$$



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⑤ i. $x(t) = e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} X(s) = \frac{1}{s+2}$, $Re\{s\} > -2$

$h(t) = u(t) \xleftrightarrow{\mathcal{L}} H(s) = \frac{1}{s}$, $Re\{s\} > 0$

$Y(s) = H(s)X(s) = \frac{1}{s}\cdot\frac{1}{s+2} = \frac{1/2}{s} - \frac{1/2}{s+2}$, ROC contains $Re\{s\} > 0$

$y(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t)$

ii. $x(t) = e^{-2t}u(t) \longleftrightarrow X(s) = \frac{1}{s+2}$, $Re\{s\} > -2$

$h(t) = e^{-3t}u(t) \longleftrightarrow H(s) = \frac{1}{s+3}$, $Re\{s\} > -3$

$Y(s) = H(s)X(s) = \frac{1}{s+2}\cdot\frac{1}{s+3} = \frac{1}{s+2} - \frac{1}{s+3}$, ROC contains $Re\{s\} > -2$

$y(t) = e^{-2t}u(t) - e^{-3t}u(t)$

iii. $x(t) = u(t) \longleftrightarrow X(s) = \frac{1}{s}$, $Re\{s\} > 0$

$h(t) = \cos(2\pi t)u(t) \longleftrightarrow H(s) = \frac{s}{s^2 + (2\pi)^2}$, $Re\{s\} > 0$

$Y(s) = H(s)X(s) = \frac{1}{s^2 + (2\pi)^2}$, ROC contains $Re\{s\} > 0$

Since $\sin(\omega_0 t)\,u(t) \longleftrightarrow \frac{\omega_0}{s^2 + \omega_0^2}$, $Re\{s\} > 0$

$y(t) = \frac{1}{2\pi}\sin(2\pi t)\,u(t)$

iv. $x(t) = \cos(2\pi t)u(t) \longleftrightarrow X(s) = \frac{s}{s^2 + (2\pi)^2}$, $Re\{s\} > 0$

$h(t) = \delta(t-1) + \delta(t-.5) \longleftrightarrow H(s) = e^{-s} + e^{-s/2}$, all s

$Y(s) = H(s)X(s) = \frac{s}{s^2 + (2\pi)^2}e^{-s} + \frac{s}{s^2 + (2\pi)^2}e^{-s/2}$, ROC contains $Re\{s\} > 0$

$y(t) = \cos(2\pi(t-1))u(t-1) + \cos(2\pi(t-.5))u(t-.5)$

v. $x(t) = \cos(2\pi t)u(t) \longleftrightarrow X(s) = \frac{s}{s^2 + (2\pi)^2}$, $Re\{s\} > 0$

$h(t) = \cos(2\pi t)u(t) \longleftrightarrow H(s) = \frac{s}{s^2 + (2\pi)^2}$, $Re\{s\} > 0$

$Y(s) = X(s)H(s) = \frac{s^2}{(s^2 + (2\pi)^2)(s^2 + (2\pi)^2)} = \frac{j/8\pi}{s + j2\pi} + \frac{1/4}{(s + j2\pi)^2} - \frac{j/8\pi}{s - j2\pi} + \frac{1/4}{(s - j2\pi)^2}$, ROC contains $Re\{s\} > 0$

$y(t) = \frac{j}{8\pi}\cdot e^{-j2\pi t}u(t) + \frac{1}{4}\cdot t e^{-j2\pi t}u(t) - \frac{j}{8\pi}\cdot e^{j2\pi t}u(t) + \frac{1}{4}\cdot t e^{j2\pi t}u(t)$

$= \frac{1}{4\pi}\sin(2\pi t)\,u(t) + \frac{1}{2}\cos(2\pi t)\,t\,u(t)$

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5v detail

$$\frac{s^2}{(s+j2\pi)(s-j2\pi)(s+j2\pi)(s-j2\pi)} = \frac{s^2}{(s+j2\pi)^2(s-j2\pi)^2} = Y(s)$$

$$Y(s) = \frac{A}{s+2j\pi} + \frac{B}{(s+j2\pi)^2} + \frac{C}{s-j2\pi} + \frac{D}{(s-j2\pi)^2}$$

$$(s+j2\pi)^2 Y(s)\Big|_{s=-j2\pi} = B = \frac{(-j2\pi)^2}{(-j2\pi-j2\pi)^2} = \frac{(2\pi)^2}{(4\pi)^2} = 1/4$$

$$(s-j2\pi)^2 Y(s)\Big|_{s=+j2\pi} = D = \frac{(+j2\pi)^2}{(+j4\pi)^2} = 1/4$$

$$\frac{d}{ds}\left[(s+2j\pi)^2 Y(s)\right]\Big|_{s=-2\pi j} = \frac{d}{ds}A(s+j2\pi) + \underbrace{\frac{d}{ds}\left[\frac{B}{1} + \frac{C(s+j2\pi)^2}{s-j2\pi} + \frac{D(s+j2\pi)^2}{(s-j2\pi)^2}\right]}_{=0 \text{ for } s=-2\pi j}$$

$$= A = \frac{d}{ds}\left[\frac{s^2}{(s-j2\pi)^2}\right] = \left.\begin{matrix}-2s^2(s-j2\pi)^{-3} \\ +2s\,(s-j2\pi)^{-2}\end{matrix}\right|_{s=-j2\pi}$$

$$= \frac{s}{(s-j2\pi)^2}\left[\frac{-2s}{s-j2\pi} + 2\right]$$

$$= \frac{-j2\pi}{(-j4\pi)^2}\cdot\left[\frac{+j4\pi}{-j4\pi} + 2\right] = \frac{-j}{-8\pi}[1]$$

$$= \boxed{\frac{+j}{8\pi} = A}$$

$$\frac{d}{ds}\left[(s-2j\pi)^2 Y(s)\right]\Big|_{s=+2\pi j} = C$$

$$= \frac{s}{(s+2\pi j)^2}\left[2 - \frac{2s}{s+2\pi j}\right] = \frac{-j}{8\pi}$$

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⑥ $$\frac{d^2y}{dt^2} + 300\frac{dy}{dt} + 2\times10^4 y(t) = 10^3 x(t).$$

$$\frac{dy}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sY(s) - y(0^-) = sY(s) - 1$$

$$\frac{d^2y}{dt^2} \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - s\,y(0^-) - \dot{y}(0^-)$$
$$= s^2Y(s) - s + 2$$

$$s^2Y(s) - s + 2 + 300sY(s) - 300 + 2\times10^4 Y(s)$$
$$= \frac{1000}{s}\ ;\ \left[\text{note } x(t) = u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}\right]$$

$$Y(s)\left[s^2 + 300s + 2\times10^4\right] = \frac{1000}{s} + s + 298$$

$$Y(s) = \frac{s^2 + 298s + 1000}{(s^2 + 300s + 2\times10^4)s}$$

$(s+100)(s+200)$

Partial Fraction Decomposition:

$$Y(s) = \frac{A}{s} + \frac{B}{s+100} + \frac{C}{s+200}$$

$$A(s+100)(s+200) + Bs(s+200) + Cs(s+100)$$
$$= s^2 + 298s + 1000$$

$s=0 \Rightarrow A(100)(200) = 1000 \rightarrow A = 1/20$

$s=-100 \Rightarrow B(-100)(100) = -18800 \rightarrow B = +47/25$

$s=-200 \Rightarrow C(-200)(-100) = -18600 \rightarrow C = -93/100$

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$$Y(s) = \frac{1/20}{s} + \frac{47/25}{s+100} - \frac{93/100}{s+200}$$

$$\therefore y(t) = \left[\frac{1}{20} + \frac{47}{25} e^{-100t} - \frac{93}{100} e^{-200t}\right] u(t)$$

$$y(t) = \left[0.05 + 1.88 e^{-100t} - 0.93 e^{-200t}\right] u(t)$$

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