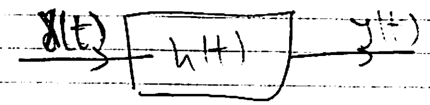
EE120: PS#2 Solutions. FALL 2014

1. $y(t) = x(t) + \alpha y(t-T)$



a) $x(t) = \delta(t)$.

$\therefore h(t) = \delta(t) + \alpha h(t-T)$

for $t < T$, $h(t) = \delta(t)$

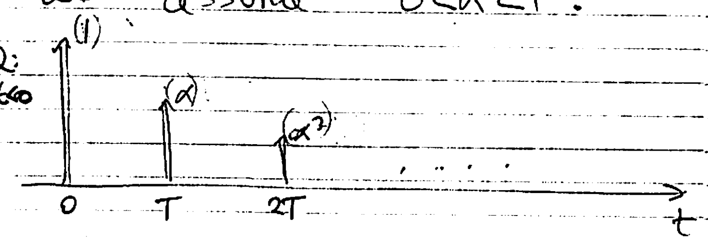
for $t < 2T$, $h(t) = \delta(t) + \alpha h(t-T) = \delta(t) + \alpha\delta(t-T)$

for $t < 3T$ $h(t) = \delta(t) + \alpha[\delta(t-T) + \alpha\delta(t-2T)]$

thus $h(t) = \sum_{n=0}^{\infty} \alpha^n \delta(t-nT)$ (continuing recursion)

If we assume $0 < \alpha < 1$:

note causal: $h(t) = 0$ for $t < 0$



b) BIBO stability requires:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

For our system,

$$\int_{-\infty}^{\infty} |h(t)| dt \Rightarrow 1 + \alpha + \alpha^2 + \alpha^3 + \cdots$$

Recall geometric series.

If $r \neq 1$, $\sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r}$

take limit $n \to \infty$:

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} ar^k = \lim_{n\to\infty} a\frac{1-r^n}{1-r}$$

If $r > 1 \Rightarrow \lim_{n\to\infty} a\frac{1-r^n}{1-r} = \infty$

If $r < 1 \Rightarrow \lim_{n\to\infty} a\frac{1-r^n}{1-r} = \frac{a}{1-r}$

$\therefore$ If $0 < \alpha < 1$,

$$1 + \alpha + \alpha^2 + \alpha^3 + \cdots = \frac{1}{1-\alpha}$$

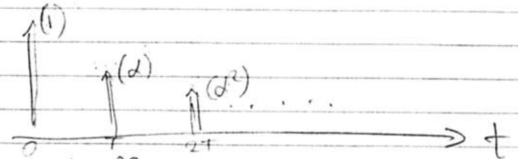
If not, then $\int |h(t)| dt$ above diverges; hence system is unstable for $\alpha \geq 1$.

c) $\xrightarrow{x(t)} \boxed{h(t)} \rightarrow \boxed{g(t)} \xrightarrow{x(t)}$.

Solve for $g(t)$ such that

$h(t) \circledast g(t) = \delta(t)$.

$h(t)$:



$$h(t) = \sum_{n=0}^{n=\infty} \alpha^n \delta(t-nT)$$

We want:

$$\int_{-\infty}^{\infty} g(\tau) \sum_{n=0}^{\infty} \alpha^n \delta(t-\tau-nT)\, d\tau = \delta(t)$$

We will assume $g(t<0) = 0$. [CAUSAL!

$$\therefore \sum_{n=0}^{\infty} \alpha^n \int g(\tau)\delta(t-nT-\tau)\, d\tau$$

$$= \sum_{n=0}^{\infty} \alpha^n g(t-nT) = \delta(t)$$

$$\therefore\ g(t) + \alpha g(t-T) + \alpha^2 g(t-2T) + \dots = \delta(t)$$

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$t=T$ $\delta(t-T)$

$\underline{t=T}$: $g(T) = 0 - \alpha g(0) - \alpha^2 g(-T) - \alpha^3 g(-2T) - \dots$

by causality of $g(t)$

$g(T) = -\alpha g(0)$.

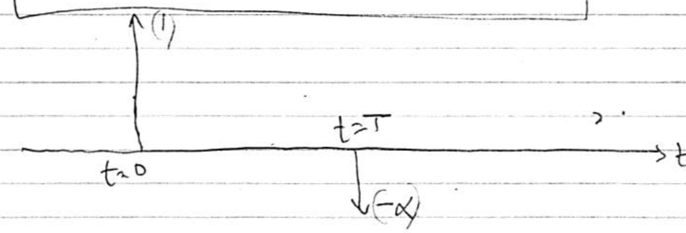
$\underline{t=2T}$: $g(2T) = 0 - \alpha g(T) - \alpha^2 g(0) - 0$

$g(2T) = \alpha^2 g(0) - \alpha^2 g(0) = 0$

$\underline{t=3T}$: $g(3T) = -\alpha g(2T) - \alpha^2 g(T) - \alpha^3 g(0) - 0$

$g(3T) = \alpha^3 g(0) - \alpha^3 g(0) = 0$.

$$\therefore\ \boxed{g(t) = \delta(t) - \alpha\,\delta(t-T).}$$



* While solving for $g(t)$ algebraically is fine, you will get full credit if you found it graphically by realizing the "flip-and-slide" rule for convolutions.

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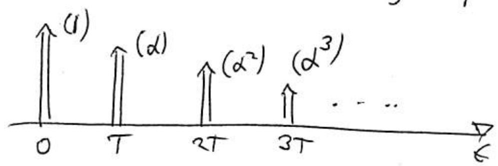
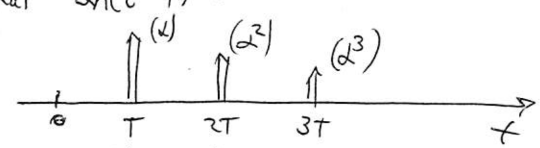
Graphical method. Find $g(t)$ which is causal, and composed of ~~ope~~ LTI operations such as scale, time shift, adding copies, etc.

$h(t)$:



desired $g(t) * h(t) = \delta(t)$

note that $\alpha h(t-T) =$



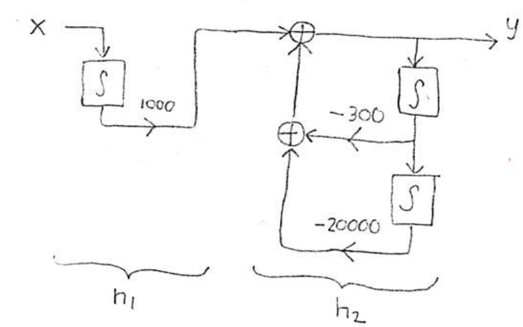
then $h(t) - \alpha h(t-T) = \delta(t)$

but $(\delta(t) - \alpha\delta(t-T)) * h(t) = h(t) - \alpha h(t-T) = \delta(t)$

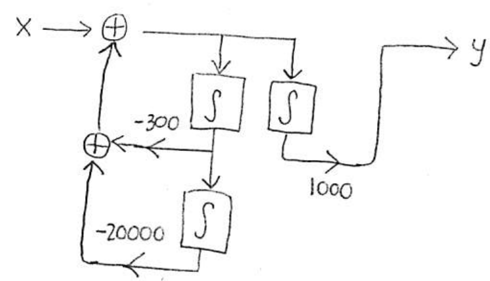
Thus $g(t) = \delta(t) - \alpha(\delta(t-T))$.

P2 a) $y'' = -300y' - 20000y + 1000x'$

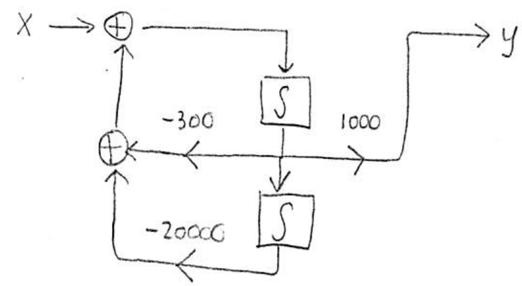
$y = \iint(-300y' - 20000y + 1000x')$



By commutativity, this is equivalent to:



which can be simplified as



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P2 (cont'd) b) $x(t) = e^{j\omega t}$, $y(t) = H(j\omega)\cdot e^{j\omega t}$

$$y''(t) + 300\,y'(t) + 20000\,y(t) = 1000\,x'(t)$$

$$H(j\omega)\cdot(j\omega)^2 e^{j\omega t} + 300H(j\omega)(j\omega)e^{j\omega t} + 20000H(j\omega)e^{j\omega t} = 1000(j\omega)e^{j\omega t}$$

$$H(j\omega)\left[(j\omega)^2 + 300j\omega + 20000\right] = 1000\,j\omega$$

$$H(j\omega) = \frac{1000\,j\omega}{(j\omega)^2 + 300j\omega + 20000}$$

So the output is $y(t) = H(j\omega)e^{j\omega t}$, where $H(j\omega)$ is

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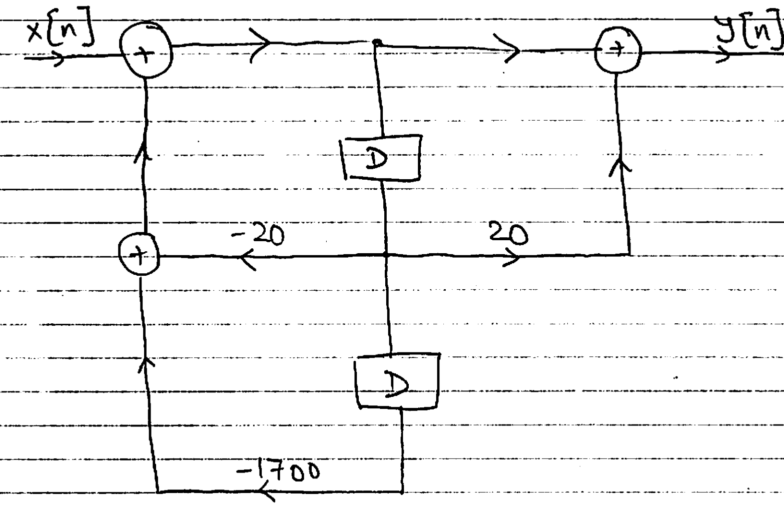
③ a)



b) General expression for $y[n]$:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[n] = \sum_{m=-\infty}^{\infty} x[n-m]h[m]$$

$$x[n] = e^{j\omega n}$$

$$\therefore\ y[n] = e^{j\omega n}\sum_{m=-\infty}^{\infty} e^{-j\omega m}\,h[m]$$

$$y[n] = e^{j\omega n}\,H(e^{j\omega})$$

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$$y[n] + 20y[n-1] + 1700y[n-2] = x[n] + 20x[n-1]$$

Substituting $y[n]$ we just found,

$$e^{j\omega n} H(e^{j\omega}) \left[1 + 20e^{-j\omega} + 1700e^{-j2\omega}\right]$$

$$= e^{j\omega n}\left[1 + 20e^{-j\omega}\right]$$

$$\therefore H(e^{j\omega}) = \frac{1 + 20e^{-j\omega}}{1 + 20e^{-j\omega} + 1700e^{-j2\omega}}$$

$$\therefore \boxed{y[n] = \frac{1 + 20e^{-j\omega}}{1 + 20e^{-j\omega} + 1700e^{-j2\omega}} e^{j\omega n}}$$

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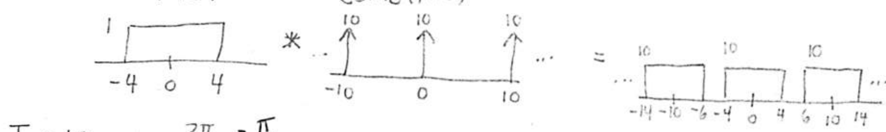
P4

a) $\Pi(t/8) * \text{comb}(.1t) =$

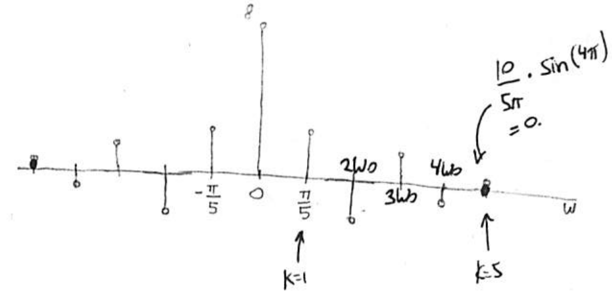


$T_0 = 10, \quad \omega_0 = \frac{2\pi}{10} = \frac{\pi}{5}$

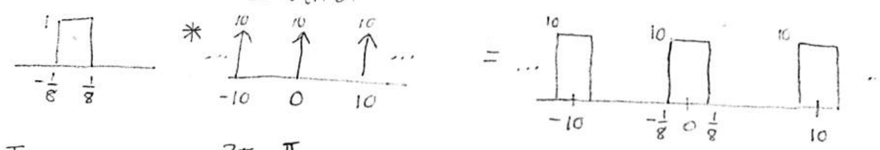
$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\frac{2\pi}{10}t}\,dt$$

$$= \frac{1}{10}\int_{-4}^{4} 10 \cdot e^{-jk\frac{\pi}{5}t}\,dt$$

$$= \frac{10}{\pi k} \cdot \sin\left(\frac{4\pi}{5}k\right)$$



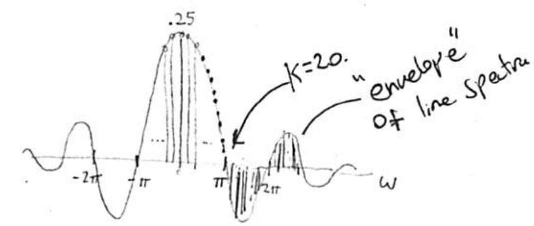
b) $\Pi(4t) * \text{comb}(.1t) =$



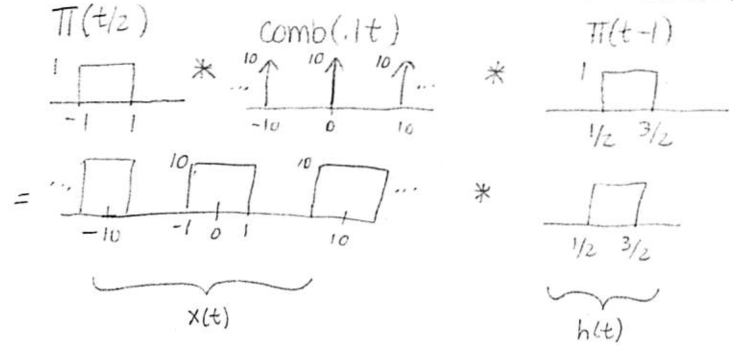
$T_0 = 10, \quad \omega_0 = \frac{2\pi}{10} = \frac{\pi}{5}$

$$a_k = \frac{1}{10}\int_{-1/8}^{1/8} 10e^{-jk\frac{\pi}{5}t}\,dt$$

$$= \frac{10}{\pi k}\sin\left(\frac{\pi}{40}k\right)$$



c) $\Pi(t/2) * \Pi(t-1) * \text{comb}(.1t) = \Pi(t/2) * \text{comb}(.1t) * \Pi(t-1)$



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f (cont'd)

(cont'd) First find a_K for $x(t)$:

$$a_K = \frac{1}{10}\int_{-1}^{1} \cdot 10 e^{-jK\frac{\pi}{5}t}\,dt$$
$$= \frac{10}{K\pi}\cdot \sin\left(\frac{\pi}{5}K\right)$$

Let $y(t) = x_1(t) * h(t)$. Think of $h(t)$ as an LTI system.
When the input to $h(t)$ is $x_1(t) = e^{j\omega t}$, the output is

$$y(t) = \int_{-\infty}^{\infty} x_1(t-\tau)\,h(\tau)\,d\tau$$
$$= \int_{-\infty}^{\infty} e^{j\omega(t-\tau)}\,\Pi(\tau-1)\,d\tau$$
$$= e^{j\omega t}\int_{1/2}^{3/2} e^{-j\omega\tau}$$
$$= \underbrace{\frac{2\cdot\sin(\omega/2)}{\omega\cdot e^{j\omega}}}_{H(j\omega)}\cdot e^{j\omega t}$$

Now, the input is a sum of exponentials given by:

$$x(t) = \sum_{K=-\infty}^{\infty} a_K\, e^{jK\frac{2\pi}{10}t}$$

So the output is then

$$y(t) = H\{x(t)\}$$
$$= H\{\textstyle\sum a_K e^{jK\frac{\pi}{5}t}\}$$
$$= \textstyle\sum a_K\, H\{e^{jK\frac{\pi}{5}t}\}$$

let $\omega = K\omega_0 = K\pi/5$

$$= \sum \underbrace{a_K\cdot\frac{2\sin(\frac{\pi}{10}K)}{\frac{\pi}{5}K\, e^{j\frac{\pi}{5}K}}}_{a_K'}\cdot e^{jK\frac{\pi}{5}t} = \sum_K a_K\,\frac{2\sin(\frac{\pi}{10}K)}{\frac{\pi}{5}K}\, e^{j\pi\frac{K}{5}(t-1)}$$

P4

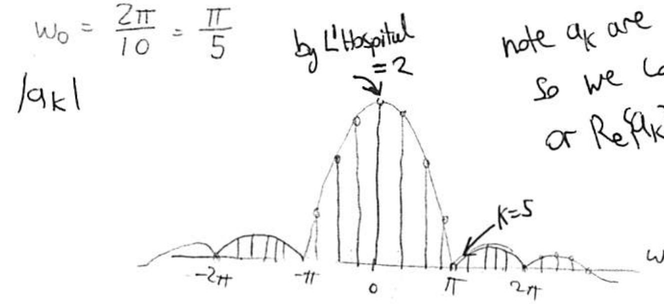
c) (cont'd)

So the Fourier series coefficients for $y(t)$ are given by:

$$a_K' = \frac{10}{K\pi}\sin\left(\frac{\pi}{5}K\right)\cdot\frac{2\sin(\frac{\pi}{10}K)}{\frac{\pi}{5}K\, e^{j\frac{\pi}{5}K}}$$
$$= \frac{100\sin(\frac{\pi}{5}K)\sin(\frac{\pi}{10}K)}{K^2\pi^2\, e^{j\frac{\pi}{5}K}}$$

$$\omega_0 = \frac{2\pi}{10} = \frac{\pi}{5}$$

note a_K are complex, So we could plot $|a_K|$, $\measuredangle a_K$ or $Re\{a_K\}$, $Im\{a_K\}$



(5)

a) $x[n] = \delta[n-1] + \delta[n-2]$; $N = 8$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} \left\{\delta[n-1] + \delta[n-2]\right\} e^{-j\frac{2\pi}{N}kn}$$

$$a_k = \frac{1}{N}\left[e^{-j\frac{2\pi}{N}k} + e^{-j\frac{4\pi}{N}k}\right]$$

$$\therefore \boxed{a_k = \frac{1}{8}\left[e^{-j\frac{\pi}{4}k} + e^{-j\frac{\pi}{2}k}\right]}$$

$$= \frac{1}{8} e^{-j\frac{3\pi k}{8}}\left[e^{+j\pi/8k} + e^{-j\pi k/8}\right] = \frac{2}{8} e^{-j\frac{3\pi 8}{k}} \cos\frac{\pi k}{8}$$

b) $x[n] = (-1)^n$; $N = 32$

$$a_k = \frac{1}{32} \sum_{n=0}^{n=31} (-1)^n e^{-jk\frac{2\pi}{32}n}$$

$$a_k = \frac{1}{32} \sum_{n=0}^{31} (-1)^n e^{-j\frac{k\pi}{16}n}$$

If $k = \pm 16, \pm 48 \Longrightarrow a_k = 1$

Else: $a_k = \frac{1}{32} \sum_{n=0}^{31} \left(-e^{-jk\frac{\pi}{16}}\right)^n$

$$\therefore a_k = \frac{1}{32} \frac{1-\left(-e^{-jk\pi/16}\right)^{32}}{1+e^{-jk\pi/16}}$$

$$a_k = \frac{1}{32} \frac{1-e^{-j2\pi k}}{1+e^{-jk\pi/16}}$$

$a_k = 0$.

$$\therefore \boxed{a_k = \begin{cases} 1 & ; \quad k = \pm 16, \pm 48, \pm 80, \dots \\ 0 & ; \quad \text{otherwise} \end{cases}}$$

note that $-1 = e^{j\pi}$, $(-1)^n = e^{j\pi n}$

$$a_k = \frac{1}{32} \sum_{n=0}^{31} e^{j\pi n} e^{-jk\pi/16n} = \frac{1}{32} \sum_{n=0}^{31} e^{j\pi n(1-k/16)}$$

$$= \frac{1}{32}\left(\sum_{n=0}^{15} e^{j\pi n(1-k/16)} + \sum_{n=16}^{31} e^{j\pi n(1-k/16)}\right)$$

$$= \frac{1}{32}\left(\sum_{n=0}^{15} e^{j\pi n(1-k/16)} + \sum_{n=0}^{15} e^{j\pi(n+16)(1-k/16)}\right)$$

now $e^{j\pi(n+16)} = e^{j\pi n} e^{j\pi 16} = e^{j\pi n}$

so

$$a_k = \frac{1}{32} \sum_{n=0}^{15} \left(e^{j\pi n(1-k/16)} + e^{j\pi\left[n(1-\frac{k}{16}) + 16 - k\right]}\right)$$

$$= \frac{1}{32} \sum_{n=0}^{15} e^{j\pi n(1-k/16)}\left[1 + e^{-j\pi k}\right] = 0 \text{ for } K \text{ odd}$$

for K even, $a_k = \frac{1}{16} \sum_{n=0}^{15} e^{j\pi n} e^{-j\pi n k/16} = 1$ for $k=16$

$= 0$ else since vectors sum to 0.

Alternate Solution

5b. $x[n] = (-1)^n$, $N=32$. Note $x[n] = \cos \pi n$

$$\Longrightarrow x[n] = \frac{1}{2}\left(e^{j\pi n} + e^{-j\pi n}\right)$$

$$a_k = \frac{1}{32}\sum_{n=0}^{31} \frac{1}{2}\left(e^{j\pi n} + e^{-j\pi n}\right) e^{-j\frac{2\pi}{32}kn}$$

recall orthogonality of $e^{j\omega_0 ln}$, that is

$$\frac{1}{N}\sum_{n=\langle N\rangle} e^{j\omega_0 ln} e^{-j\omega_0 mn} = \frac{1}{N}\sum e^{j\frac{2\pi}{N}ln} e^{-j\frac{2\pi}{N}mn}$$

$$= \begin{cases} 0, & m \neq l \\ 1, & l = m \end{cases}$$

regrouping & rewriting a_k,

$$a_k = \frac{1}{2}\cdot\frac{1}{32}\sum_n \left(e^{j\frac{2\pi}{32}\cdot 16\cdot n} + e^{-j\frac{2\pi}{32}\cdot 16\cdot n}\right) e^{-j\frac{2\pi}{32}kn}$$

$$= \frac{1}{2}\cdot\frac{1}{32}\sum_n \left(e^{-j\frac{2\pi}{32}(k-16)n} + e^{-j\frac{2\pi}{32}(k+16)n}\right)$$

$\Longrightarrow a_k = 0$ unless $k = \pm 16, \pm\cancel{48}, \ldots$

$= 1$ for $k = \pm 16, \pm\cancel{48}, \ldots$

Important note: $a_{16}\, e^{j\frac{2\pi}{32}\cdot 16n} = a_{16}\, e^{j\frac{2\pi}{32}\cdot(16+32)n}$ k k

DTFS for $N=32$ only has 32 basis functions

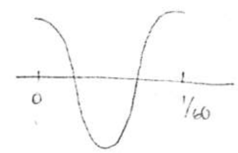
$a_0 \cdots a_{31}$ ~~Xlat~~

P6

a) $x_1(t) = \cos(120\pi t)$ $T_0 = \frac{1}{60}$ $\omega_0 = 120\pi$

$$= \frac{1}{2}e^{j120\pi t} + \frac{1}{2}e^{-j120\pi t}$$

$$= \sum_{k=-\infty}^{\infty} a_k\, e^{jk\frac{2\pi}{1/60}t}$$



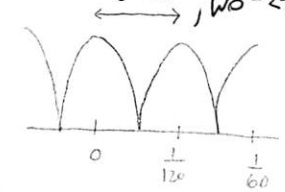
$$a_k = \begin{cases} \frac{1}{2}, & k=1 \text{ or } k=-1 \\ 0 & \text{else} \end{cases}$$

$$\text{Power} = \frac{1}{T}\int_T \cos^2(120\pi t)\,dt$$

$$= 60\cdot\int_0^{1/60} \frac{1+\cos(240\pi t)}{2}\,dt$$

$$= \frac{1}{2} = |a_1|^2 + |a_{-1}|^2 = \frac{1}{2} \checkmark$$

$T_0 = \frac{1}{120}$, $\omega_0 = 240\pi$



b) $x_2(t) = |\cos(120\pi t)|$

$$a_k = 120\int_{-1/240}^{1/240} \cos(120\pi t)\cdot e^{-jk240\pi t}\,dt$$

$$= 60\int_{-1/240}^{1/240} e^{j(120-240k)\pi t}\,dt + 60\int_{-1/240}^{1/240} e^{-j(120+240k)\pi t}$$

$$= \frac{\sin((\frac{1}{2}-k)\pi)}{(1-2k)\pi} + \frac{\sin((\frac{1}{2}+k)\pi)}{(1+2k)\pi}$$

$$= \frac{(-1)^k}{\pi(1-2k)} + \frac{(-1)^k}{\pi(1+2k)}$$

Note that the power is the same as in part a

Power $= \frac{1}{2}$

P6

c) The transfer function of $h(t)$ is

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} e^{-120\pi t} \cdot e^{-j\omega t}\, dt$$

$$= \frac{1}{120\pi + j\omega}$$

$$x_3(t) = H\{x_1(t)\}$$

$$= H\{\sum_K a_K e^{jK120\pi t}\}$$

$$= \sum_K a_K H\{e^{jK120\pi t}\}$$

$$= \sum_K \underbrace{a_K \frac{1}{120\pi + j\cdot 120\pi K}}_{a_K'} e^{jK120\pi t}$$

So

$$a_K' = \begin{cases} \frac{1}{2}\cdot\frac{1}{120\pi + j\cdot 120\pi}, & K=1 \\ \frac{1}{2}\cdot\frac{1}{120\pi - j\cdot 120\pi}, & K=-1 \\ 0 & \text{else} \end{cases}$$

By Parseval's thm, the power is:

$$\text{Power} = \sum |a_K'|^2$$

$$= \frac{1}{4}\cdot\frac{1}{2\cdot(120\pi)^2} + \frac{1}{4}\cdot\frac{1}{2(120\pi)^2}$$

$$= \frac{1}{4(120\pi)^2}$$

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P6

d) $x_4(t) = H\{x_2(t)\}$

$$= \sum_K a_K \cdot H\{e^{jK240\pi t}\}$$

$$= \sum_K \underbrace{a_K \cdot \frac{1}{120\pi + j\cdot K240\pi}}_{a_K'} \cdot e^{jK240\pi t}$$

So

$$a_K' = \frac{1}{120\pi + j\,240K\pi}\cdot\left[\frac{(-1)^K}{\pi(1-2K)} + \frac{(-1)^K}{\pi(1+2K)}\right]$$

$$\text{Power} = \sum |a_K'|^2$$

$$= \sum_{K=-\infty}^{\infty} \frac{1}{(120\pi)^2 + (240\pi K)^2}\cdot\frac{1}{\pi^2}\cdot\left|\frac{1}{1-2K} + \frac{1}{1+2K}\right|^2$$

no simple closed form, so this is ok.