1 Derivation of the Z transform

The Z transform is the discrete time analog of the Laplace transform. As for the LT, the ZT allows modelling of unstable systems as well as initial and final values. (The DTFT can not be applied if the unit circle $e^{j\omega T}$ is not part of the region of convergence.)

A continuous time signal $x(t)$ is sampled in time at rate T. The time sampled signal is given by:

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

The Laplace transform of $x_\delta(t)$ is given by:

$$\begin{aligned} X_\delta(s) &= \int_{t=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)e^{-st}dt \\ &= \sum_{n=-\infty}^{\infty} x(nT) \int_{t=-\infty}^{\infty} \delta(t-nT)e^{-st}dt \\ &= \sum_{n=-\infty}^{\infty} x(nT)e^{-snT}, \end{aligned}$$

where summation and integration were swapped, and the sifting property of the delta function was used. By substituting $x(nT) = x[n]$, and $z = e^{sT}$, we get the bilateral Z transform of $x[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$