An interesting issue arises when a windowed signal is sampled exactly at the edges. Referring to Figure 1, consider a rectangular window $w(t) = \Pi(t/2) = u(t+1) - u(t-1)$, and sampling rate $T_s = 1.0$ sec. Let $x(t) = 1$, to consider effects of the window. (Note that in problem set 4, we did not sketch the sampled and windowed signal in time.)

Let the sampling function be $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-n)$.
Then

$$\begin{aligned} x_\delta(t) &= w(t) * p(t) & (1)\\ &= \Pi(t/2)[\delta(t+1) + \delta(t) + \delta(t-1)] & (2)\\ &= \delta(t+1)u(t+1) + \delta(t) + \delta(t-1)(1-u(t-1))] & (3)\\ &= 0.5\delta(t+1) + \delta(t) + 0.5\delta(t-1) & (4) \end{aligned}$$

if we take $u(t=0) = 0.5$. We can show this is the case by calculating $X_\delta(j\omega)$ and then using the inverse Fourier transform.

Calculate Fourier transforms:

$$w(t) = \Pi(t/2) \rightarrow W(j\omega) = \frac{2\sin\omega}{\omega}$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-n) \rightarrow P(j\omega) = 2\pi \sum_{k\ =\ -\infty}^{\infty} \delta(\omega - k2\pi)$$

The sampled spectrum is obtained from convolution in frequency, with

$$X_\delta(j\omega) \;=\; \frac{1}{2\pi} W(j\omega) * P(j\omega) = \frac{1}{2\pi} W(j\omega) * 2\pi \sum_{k\ =\ -\infty}^{\infty} \delta(\omega - k2\pi)$$

The spectrum for the sampled window is then:

$$X_\delta(j\omega) \;=\; \sum_{k=-\infty}^{\infty} W(j(\omega - 2\pi k)$$

Several frequency points are easy to calculate: $\omega = 0, 2\pi, 4\pi, ...$, $\omega = \pi, 3\pi, 5\pi, ...$, and $\omega = \frac{\pi}{2}, \frac{3\pi}{2}, ...$:

$$X_\delta(j2\pi n) \;=\; \sum_{k=-\infty}^{\infty} W(j2\pi(n-k)) = 2\delta[n-k]$$

$$X_\delta(j(2n+1)\pi) \;=\; \sum_{k=-\infty}^{\infty} W(j\pi(2n+1-2k)) = \frac{2\sin\pi(2n+2k+1)}{\pi(2n+2k+1)} \;=\; 0$$

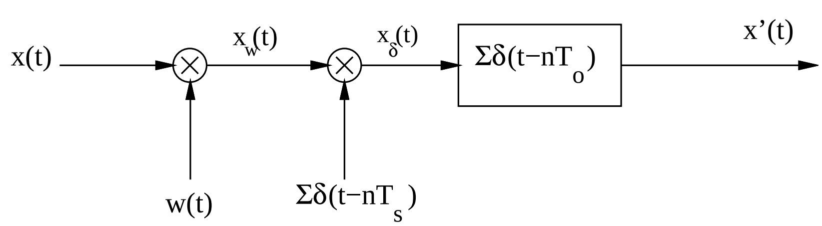

Figure 1: Block diagram of DFT processing steps.

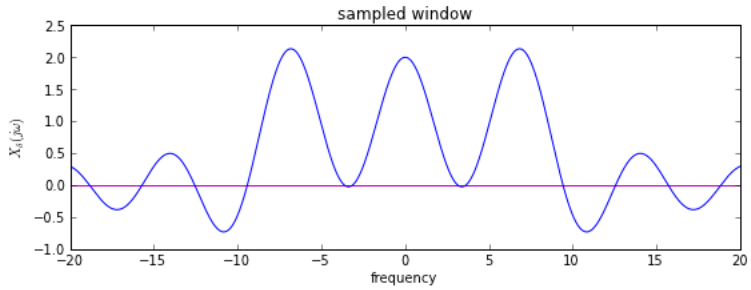

Figure 2: Superposition of 3 sinc functions centered at $-2\pi, 0, 2\pi$.

We know that $X_\delta(j\omega)$ is periodic with period 2π. Since sinc() is even,

$$X_\delta(\pm j\frac{\pi}{2}) = X_\delta(\pm j\frac{3\pi}{2}) = X_\delta(\pm j\frac{5\pi}{2}) = ...$$

Adding up all the 'aliased" copies of the sincs at $\omega = j\frac{\pi}{2}$, we get:

$$X_\delta(j\frac{\pi}{2}) \;=\; \sum_{k=-\infty}^{\infty} W(j(2\pi k - \frac{\pi}{2})) \tag{5}$$

For a single sinc:

$$W(j(2\pi k - \frac{\pi}{2})) = \frac{2\sin[2\pi k - \frac{\pi}{2}]}{2\pi k - \frac{\pi}{2}} = \frac{2\sin[2\pi k - \frac{\pi}{2}]}{\frac{\pi}{2}(4k-1)} = \frac{4}{\pi}[\frac{-1}{4k-1}]$$

For $k = 0, 1, 2, 3, ...$ we get

$$\frac{4}{\pi}(1, \frac{-1}{3}, \frac{-1}{7}, \frac{-1}{11}, ...)$$

For $k = -1, -2, -3, ...$ we get

$$\frac{4}{\pi}(\frac{1}{5}, \frac{1}{9}, \frac{1}{13}, ...)$$

Using the samples at odd multiples of $\frac{\pi}{2}$ from eqn. (5), we get:

$$X_\delta(j\frac{\pi}{2}) \;=\; \sum_{k=-\infty}^{\infty} \frac{4}{\pi}[\frac{-1}{4k-1}] = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{-1^n}{2n+1} \tag{6}$$

Conveniently, eqn. (6) is just the Taylor series for $\tan^{-1}(1)$, i.e.:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}, ...$$

.

Thus $X_\delta(j\frac{\pi}{2}) \;=\; \frac{4}{\pi}\frac{\pi}{4} = 1$

Perhaps surprisingly, it can be shown that

$$X_\delta(j\omega) \;=\; \cos(\omega) + 1.$$

This spectrum is obtained by the Fourier transform of $x_\delta(t)$:

$$x_\delta(t) = 0.5\delta(t+1) + \delta(t) + 0.5\delta(t-1) \rightarrow X_\delta(j\omega) = 0.5e^{j\omega} + 1 + 0.5e^{-j\omega} = \cos(\omega) + 1$$

A sum of shifted sincs tends to a sinusoid as suggested by the spectrum shown in Fig. 2.