Notes 27 largely plagiarized by %khc

This material is optional. It will be a preview for those of you who wish to take ee123.

1 Introduction

Previously, the discrete Fourier transform was introduced as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \tag{1}$$

and its inverse transform as

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N} \tag{2}$$

These transforms, if evaluated explicitly, would result in $O(n^2)$ operations. However, by taking advantage of the properties of $e^{j2\pi/N}$, the fast Fourier transform (FFT) reduces the number of operations to $O(n \log_2 n)$. If this reduction appears trivial, consider the case where n is 1024; explicit evaluation is two orders of magnitude slower than the FFT.

2 Mathematical Derivation of Time Decimation Algorithm

One of the more useful implementations of the FFT requires N to be a power of two. Given this restriction, we search for a "divide-and-conquer" strategy that lets us divide an N point FFT into two $\frac{N}{2}$ point FFTs, since smaller problems are always easier to work on than larger ones. Once these two smaller FFTs have been performed, their results are then appropriately combined to give a solution for the original FFT.

Using the notation previously discussed in notes25, we can write the N point DFT as

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} \tag{3}$$

where $W_N = e^{-j2\pi/N}$. W_N can be interpreted as the first of the Nth roots of unity, the other roots being the other $N - 1$ powers of W_N.

If the sum above is divided into two separate sums, one of the even components of $x[n]$ and the other of the odd components of $x[n]$ (we are fortunate that N is even), the DFT then becomes

$$\begin{aligned} X[k] &= \sum_{n=0}^{N/2-1} x[2n] W_N^{2nk} + \sum_{n=0}^{N/2-1} x[2n+1] W_N^{(2n+1)k} && (4) \\ &= \sum_{n=0}^{N/2-1} x[2n] W_N^{2nk} + W_N^k \sum_{n=0}^{N/2-1} x[2n+1] W_N^{2nk} && (5) \end{aligned}$$

This division in time is also referred to as "decimation in time".

However, squaring W_N gives the first of the $\frac{N}{2}$th roots of unity. Symbolically,

$$\begin{aligned} W_N^2 &= (e^{-j2\pi/N})^2 && (6) \\ &= (e^{-j2\pi 2/N}) && (7) \\ &= (e^{-j2\pi/(N/2)}) && (8) \\ &= W_{N/2}^n && (9) \end{aligned}$$

The DFT then simplifies to

$$X[k] = \sum_{n=0}^{N/2-1} x[2n] W_{N/2}^{nk} + W_N^k \sum_{n=0}^{N/2-1} x[2n+1] W_{N/2}^{nk} \tag{10}$$

But the first sum is the $\frac{N}{2}$ point FFT of the even components of $x[n]$ and the second sum is W_N^k multiplied by the $\frac{N}{2}$ point FFT of the odd components of $x[n]$. We could stop here, having derived an expression for the N point FFT in terms of the sum of two $\frac{N}{2}$ point FFTs, but there is a further simplification that we can do.

For the last $\frac{N}{2}$ terms corresponding to $k = \frac{N}{2}$ to $k = N-1$, we start with Equation (10). Substituting $k = k' + \frac{N}{2}$, with k' ranging from 0 to $\frac{N}{2} - 1$:

$$X[k' + \frac{N}{2}] = \sum_{n=0}^{N/2-1} x[2n] W_{N/2}^{n(k'+N/2)} + W_N^{k'+N/2} \sum_{n=0}^{N/2-1} x[2n+1] W_{N/2}^{n(k'+N/2)} \tag{11}$$

Noting that

$$W_{N/2}^{n(N/2)} = 1^n = 1 \tag{12}$$

and that

$$W_N^{N/2} = (e^{-j2\pi/N})^{N/2} = e^{-j\pi} = -1 \tag{13}$$

we then simplify to obtain

$$X[k' + \frac{N}{2}] = \sum_{n=0}^{N/2-1} x[2n] W_{N/2}^{nk'} - W_N^{k'} \sum_{n=0}^{N/2-1} x[2n+1] W_{N/2}^{nk'} \tag{14}$$

The first sum is the $\frac{N}{2}$ point FFT of the even components of $x[n]$ and the second sum is $-W_N^{k'}$ multiplied by the $N/2$ point FFT of the odd components of $x[n]$. We now have found a formula for the last $N/2$ terms corresponding to $k = \frac{N}{2}$ to $k = N-1$.

Equations (10) and (14) together constitute the FFT. This is the pinnacle of life as you know it in ee120.

3 Implementation of Time Decimation Algorithm

In Figure 1, an 8 point FFT has been implemented with adders and multipliers. In part (a), we expand the 8 point FFT into two 4 point FFTs, along with machinery to reconstruct the 8 point FFT from its two smaller components. The upper FFT takes the even components of $x[n]$ as input, and the lower one takes the odd components. In part (b), we expand the 4 point FFT into two 2 point FFTs, and in part (c), that 2 point FFT reduces to a tiny package of lines.

In part (d), we put everything back together. This artful maze of is sometimes referred to as the "butterfly", although it looks more like a mutated spider to me. Your mileage may vary.

Note that the input to the 8 point FFT is not ordered as you would think. For an interesting method of determining what that order should be, consider the fourth input, $x[6]$. If we write 6 in binary, we would obtain 110. Reversing those bits gives 011, which is the binary representation of 3. In general, the order of the input is the bit-reversal of its binary representation.

Note that even though there are n operations at every stage in the butterfly, there are only $\log_2 n$ stages. This gives an order of growth of $O(n \log_2 n)$ for the FFT.

4 Summary

- Appropriately massaging the DFT produces the FFT.
- We have developed the time decimation version of the FFT:

$$X[k] = \sum_{n=0}^{N/2-1} x[2n] W_{N/2}^{nk} + W_N^k \sum_{n=0}^{N/2-1} x[2n+1] W_{N/2}^{nk} \tag{15}$$

for $k = 0, 1, \ldots, N-1$. In other words, the N point FFT is just the sum of the $\frac{N}{2}$ point FFT of the even samples of $x[n]$ and an appropriately scaled $\frac{N}{2}$ point FFT of the odd samples of $x[n]$.

- Properties of W_N allow us to rewrite the above as

$$X[k] = \sum_{n=0}^{N/2-1} x[2n]W_{N/2}^{nk} + W_N^k \sum_{n=0}^{N/2-1} x[2n+1]W_{N/2}^{nk} \text{ for } k = 0, 1, \ldots \frac{N}{2} - 1 \quad (16)$$

$$X[k' + \frac{N}{2}] = \sum_{n=0}^{N/2-1} x[2n]W_{N/2}^{nk'} - W_N^{k'} \sum_{n=0}^{N/2-1} x[2n+1]W_{N/2}^{nk'} \text{ for } k' = 0, 1, \ldots, \frac{N}{2} - 1 \quad (17)$$

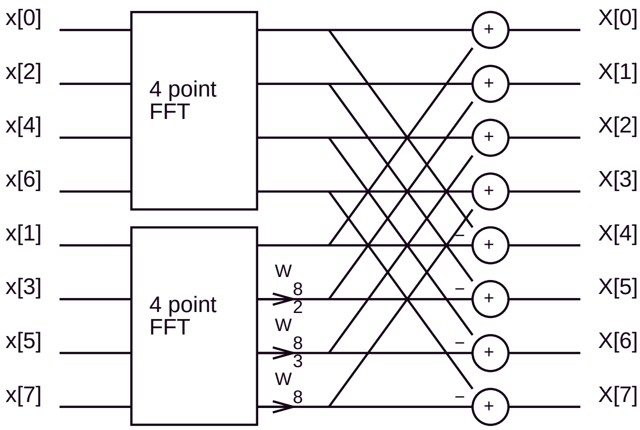
  The first equation gives $X[k]$ for $k = 0, 1, \ldots, \frac{N}{2} - 1$, and the second equation gives $X[k]$ for $k = \frac{N}{2}, \frac{N}{2} + 1, \ldots, N-1$.

- The order of growth of this algorithm is $O(n \log_2 n)$.

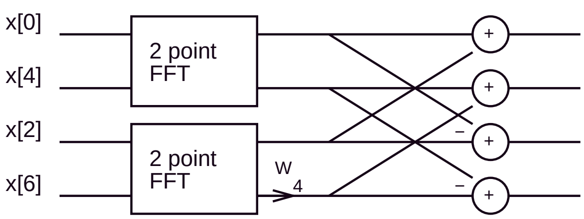
References

[1] T. H. Cormen, C. E. Leiserson, and R. L. Rivest. Introduction to Algorithms. Cambridge: MIT Press, 1990.

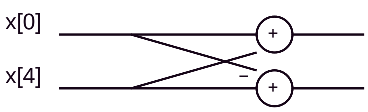
[2] G. Strang. Linear Algebra and Its Applications. San Diego: Harcourt, Brace, Jovanovich, 1988.



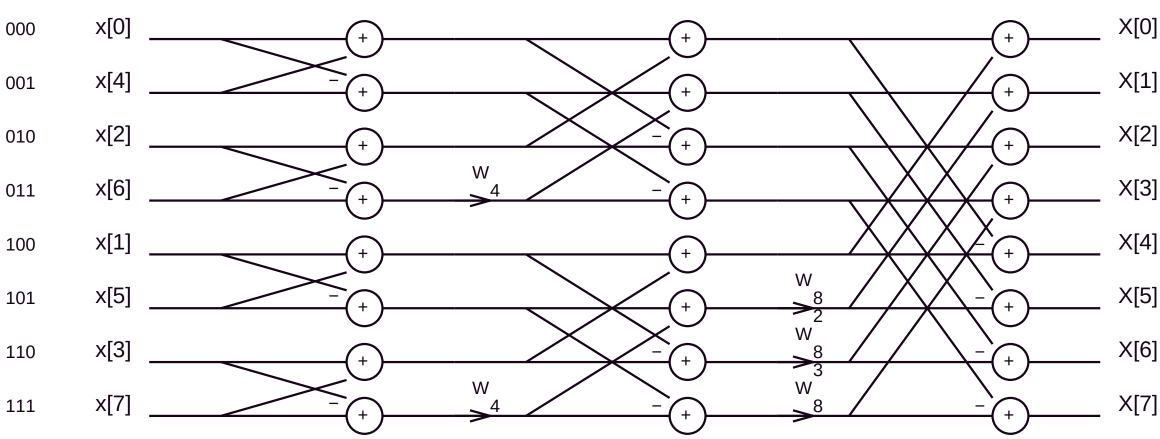
(a) The expansion of the 8 point FFT.



(b) The expansion of the 4 point FFT.



(c) The expansion of the 2 point FFT.



(d) The whole mess.

Figure 1: Implementing the FFT.