Notes 09 largely plagiarized by %khc

1 On the Relationship Between ω and f

We have been studying the Fourier transform as a function of angular frequency ω. However, since $\omega = 2\pi f$, we can also express the Fourier transform in terms of plain old frequency f:

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t}dt \end{aligned}$$

In other words, $X(\omega) = X(f)$, so graphs of $X(\omega)$ and $X(f)$ will have the same shape or form, but their x axis scaling will differ by a factor of 2π.

The inverse tranform looks slightly different though. We begin with the inverse transform in terms of ω and substitute:

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \\ &= \int_{-\infty}^{\infty} X(2\pi f)e^{j2\pi f t}df \\ &= \int_{-\infty}^{\infty} X(f)e^{j2\pi f t}df \end{aligned}$$

Note the equality of $X(\omega)$, $X(2\pi f)$, and $X(f)$ [the first is the same as the second, and the second is the same as the third, by definition].

How do the rest of the properties change?

Convolution in time is multiplication in the frequency domain in terms of both ω and f.

$$\begin{aligned} x(t) * y(t) &\leftrightarrow X(\omega)Y(\omega) \\ x(t) * y(t) &\leftrightarrow X(f)Y(f) \end{aligned}$$

However, multiplication in time will be convolution in the frequency domain, but there will be an additional factor of 2π depending on the frequency variable. Once again, we begin in terms of ω and substitute:

$$\begin{aligned} x(t)y(t) &\leftrightarrow \frac{1}{2\pi}X(\omega) * Y(\omega) \\ &\leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega - \nu)Y(\nu)d\nu \\ &\leftrightarrow \int_{-\infty}^{\infty} X(2\pi f - 2\pi\lambda)Y(2\pi\lambda)d\lambda \\ &\leftrightarrow \int_{-\infty}^{\infty} X(f - \lambda)Y(\lambda)d\lambda \end{aligned}$$

where we have made the change of variables $2\pi\lambda = \nu$, and taken advantage of the fact that $X(\omega)$, $X(2\pi f)$, and $X(f)$ are equivalent.

The duality property also appears different. Originally, we had:

$$\begin{aligned} x(t) &\leftrightarrow X(\omega) \\ X(t) &\leftrightarrow 2\pi x(-\omega) \end{aligned}$$

This comes about from considering $\mathcal{F}[X(-t)]$:

$$\begin{aligned}
\mathcal{F}[X(-t)] &= \int_{-\infty}^{\infty} X(-t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} X(-t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} X(t')e^{j\omega t'}dt' \\
&= 2\pi\frac{1}{2\pi}\int_{-\infty}^{\infty} X(t')e^{j\omega t'}dt' \\
&= 2\pi x(\omega)
\end{aligned}$$

Applying the time reversal property to this result gives the duality property as desired. Now, in terms of f, we will end up instead with:

$$\begin{aligned}
x(t) &\leftrightarrow X(f) \\
X(t) &\leftrightarrow x(-f)
\end{aligned}$$

Exercise Verify this.

This means that certain transforms will *appear* to change:

$$\begin{aligned}
1 &\leftrightarrow \delta(f) \\
u(t) &\leftrightarrow \frac{1}{j2\pi f} + \frac{\delta(f)}{2} \\
e^{j2\pi f_0 t} &\leftrightarrow \delta(f - f_0) \\
\sin 2\pi f_0 t &\leftrightarrow \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)] \\
\cos 2\pi f_0 t &\leftrightarrow \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)] \\
\sum_{k=-\infty}^{\infty} a_k e^{jk2\pi f_0 t} &\leftrightarrow \sum_{k=-\infty}^{\infty} a_k \delta(f - kf_0)
\end{aligned}$$

But actually they do not. What they all rely on is that fact that:

$$\delta(\omega) = \delta(2\pi f) = \frac{1}{2\pi}\delta(f)$$

This factor of $\frac{1}{2\pi}$ runs around and kills off the 2πs that show up all over the place when we write transforms in terms of ω.

Exercise Verify these transforms. Convince yourself that the transforms are not truly "different".

Exercise Verify that $\frac{1}{\pi t}$ [the hilbert transform] has transform $-j\text{sgn}\,(f)$. [Short way: make substitution $\omega = 2\pi f$. Long way: start from sgn $(t) \leftrightarrow \frac{2}{j\omega}$, convert to f by substituting for ω, and then use the f version of the duality property. Really long way: stick $\frac{1}{\pi t}$ into the fourier transform integral. Scream wildly and jump out the nearest window.]

Other than the cosmetic change of $\omega = 2\pi f$, the time shift, modulation, and differentiation properties will look the same. The integration property will appear to be different, but upon closer examination, nothing important will have changed [the integration property was derived by considering the convolution of $x(t)$ with $u(t)$ (check this), and $u(t)$ looks different as above].

It is useful to be sufficiently adroit with both ω and f domain representations, as certain books, papers, and classes will arbitrarily choose one or the other.