Notes 06 largely plagiarized by %khc

1 Fourier Series Revisited

Parts of this section are recycled from notes05. i feel this interpretation is sufficiently important that it bears repeating.

Any signal can be decomposed into a sum of appropriately scaled basis functions. Recall linear algebra if you ever took it. A basis is a linearly independent set of vectors that spans a vector space; the number of vectors in that linearly independent set is the dimension of the vector space. We can think of a vector space of functions and imagine a set of basis functions for that space. We could try $\{1, t, t^2, t^3, \ldots\}$— this is one basis that you have seen already (think Taylor series). But for the Fourier series, we use the complex exponentials, each an integer multiple of some fundamental frequency.

Of course, we could always use another set of basis functions, but complex exponentials will suffice for now.

One good thing about making the basis functions orthogonal is that the presence or absence of a given basis function does not affect the contribution of the other basis functions, so we can add or delete the contribution from a given basis function without changing the coefficients associated with the other basis functions. That means all we have to do is calculate a given a_k once and only once.

Another thing to keep in mind— Fourier series are valid only for periodic signals. The periodicity in time forces the FS coefficients to be discrete in frequency. Periodicity in one domain forces discreteness in the other domain. This fact will show up again later on down the road when we talk about the discrete-time Fourier transform (DTFT) and the discrete Fourier transform (DFT).

How in the world does one remember the Fourier series analysis integral? Well, in addition to writing it a billion times, one way is to remember the derviation. We multiplied took the complex conjugate of the basis function, multiplied it by x(t), and integrated over one period in time. The $\frac{1}{T}$ comes about because the basis functions have not been normalized to unity.

At this point in time, it might be useful to familiarize yourself with Table 4.2 on page 224 of your textbook (if you've never opened your textbook, then you might want to get your $70 worth). Table 4.3 on the next page also contains some useful information in the form of the Fourier series for some common functions.

2 Some Fourier Series

square wave: period T; over one period: unity for $|t| < T_1$, zero elsewhere in that period (illustrated in Figure 1).

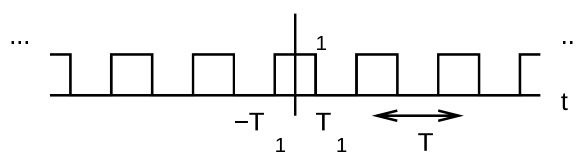

Figure 1: A square wave.

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\,dt \\
&= -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} \\
&= \frac{2}{k\omega_0 T}\frac{1}{2j}\left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right] \\
&= \frac{2}{k\omega_0 T}\sin k\omega_0 T_1 \\
&= \frac{1}{k\pi}\sin 2k\pi\frac{T_1}{T}
\end{aligned}
$$

This $\frac{\sin x}{x}$ form is called a sinc. Note that the sinc is not undefined at $x = 0$ (use l'Hôpital's rule).

comb: $\sum_{n=-\infty}^{\infty} \delta(at - nT)$.

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(at) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(at) dt \\ &= \frac{1}{|a|T}\int_{-T/2}^{T/2} \delta(t) dt \\ &= \frac{1}{|a|T} \end{aligned}$$

Note that the period of this function is not T.
Exercise Why? What is the actual period?

$\cos\omega_0 t$: $\cos\omega_0 t = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$

$$a_k = \begin{cases} \frac{1}{2} & \text{for } k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$\sin\omega_0 t$: $\sin\omega_0 t = \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}]$

$$a_k = \begin{cases} \frac{1}{2j} & \text{for } k = 1 \\ -\frac{1}{2j} & \text{for } k = -1 \\ 0 & \text{otherwise} \end{cases}$$

1: DC only.

$$a_k = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

A detailed discussion of the FS properties will not be made here. Instead, i choose to reserve this discussion until later on when we talk about the Fourier transform. Suffice to say that even those *i* choose to do that, you are still responsible for the properties on your midterm examination. Table 4.2 contains the relevant information on FS properties.

Please note that the integration property is valid only for signals that have no DC component [you cannot integrate a signal with DC in it from $t = -\infty$ and expect the integral to be bounded].

However, symmetry properties will be discussed in the next section.

3 FS Symmetry Properties

In previous math classes, you were introduced to the following two expressions for the even and odd portions of a function, as well as two expressions for the real and imaginary parts of a function:

$$\begin{aligned} \mathcal{E}v\ x(t) &= \frac{1}{2}x(t) + \frac{1}{2}x(-t) \\ \mathcal{O}d\ x(t) &= \frac{1}{2}x(t) - \frac{1}{2}x(-t) \\ \mathcal{R}e\ x(t) &= \frac{1}{2}x(t) + \frac{1}{2}x^*(t) \\ \mathcal{I}m\ x(t) &= \frac{1}{2}x(t) - \frac{1}{2}x^*(t) \end{aligned}$$

However, these notions of evenness and oddness are only for real valued functions. For complex valued functions, we extend the definitions to:

$$\begin{aligned}
\mathcal{CS}\ x(t) &= \frac{1}{2}x(t) + \frac{1}{2}x^*(-t) \\
\mathcal{CAS}\ x(t) &= \frac{1}{2}x(t) - \frac{1}{2}x^*(-t)
\end{aligned}$$

where $\mathcal{CS}$ stands for *conjugate symmetric* and $\mathcal{CAS}$ stands for *conjugate antisymmetric.* Note the presence of the complex conjugates on the second term in each expression.

Using our standard definition for FS coefficients, we find:

$$\begin{aligned}
x(t) &\leftrightarrow X_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
x^*(t) &\leftrightarrow \frac{1}{T}\int_T x^*(t)e^{-jk\omega_0 t}dt \\
&\quad [\frac{1}{T}\int_T x(t)e^{jk\omega_0 t}dt]^* \\
&\quad [\frac{1}{T}\int_T x(t)e^{-j(-k)\omega_0 t}dt]^* \\
&\quad X^*_{-k} \\
x(-t) &\leftrightarrow \frac{1}{T}\int_T x(-t)e^{-jk\omega_0 t}dt \\
&\quad \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(-t)e^{-jk\omega_0 t}dt \\
&\quad -\frac{1}{T}\int_{\frac{T}{2}}^{-\frac{T}{2}} x(\tau)e^{jk\omega_0 \tau}d\tau \\
&\quad \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)e^{-j(-k)\omega_0 \tau}d\tau \\
&\quad \frac{1}{T}\int_T x(\tau)e^{-j(-k)\omega_0 \tau}d\tau \\
&\quad X_{-k} \\
x^*(-t) &\leftrightarrow \frac{1}{T}\int_T x^*(-t)e^{-jk\omega_0 t}dt \\
&\quad [\frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x(-t)e^{jk\omega_0 t}dt]^* \\
&\quad [-\frac{1}{T}\int_{\frac{T}{2}}^{-\frac{T}{2}} x(\tau)e^{-jk\omega_0 \tau}d\tau]^* \\
&\quad [\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(\tau)e^{-jk\omega_0 \tau}d\tau]^* \\
&\quad [\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0 \tau}d\tau]^* \\
&\quad X^*_k
\end{aligned}$$

Applying the above mess to the relationships for conjugate symmetric, conjugate antisymmetric, real, and imaginary, we find:

$$\mathcal{CS}\ x(t) = \frac{1}{2}x(t) + \frac{1}{2}x^*(-t) \quad \leftrightarrow \quad \frac{1}{2}X_k + \frac{1}{2}X^*_k = \mathcal{R}e\ X_k$$

$$\begin{aligned}
\mathcal{CAS}\ x(t) = \frac{1}{2}x(t) - \frac{1}{2}x^*(-t) &\quad\leftrightarrow\quad \frac{1}{2}X_k - \frac{1}{2}X_k^* = \mathcal{I}m\ X_k \\
\mathcal{R}e\ x(t) = \frac{1}{2}x(t) + \frac{1}{2}x^*(t) &\quad\leftrightarrow\quad \frac{1}{2}X_k + \frac{1}{2}X_{-k}^* = \mathcal{CS}\ X_k \\
\mathcal{I}m\ x(t) = \frac{1}{2}x(t) - \frac{1}{2}x^*(t) &\quad\leftrightarrow\quad \frac{1}{2}X_k - \frac{1}{2}X_{-k}^* = \mathcal{CAS}\ X_k
\end{aligned}$$

In other words, real in one domain gives conjugate symmetric in the other domain; imaginary in one domain gives conjugate antisymmetric in the other domain.

4 A Filtering Problem

Given the input $x(t)$ and filter specification $H(\omega)$ for an LTI system in Figure 2, determine the output.

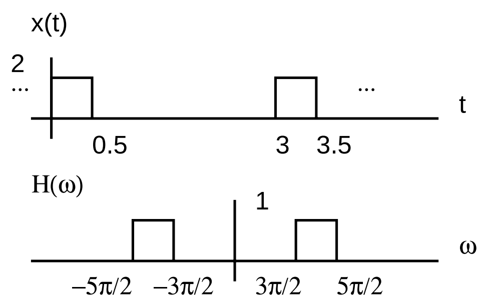

Figure 2: A filtering problem.

How do we approach this problem? Well, the input is periodic with period 3, so this suggests finding the Fourier series of the input. We can then apply the fact that $e^{j\omega t}$ is an eigenfunction for LTI systems, so the output will be each component of the input multiplied by some complex constant that we can get from $H(\omega)$.

Let's first determine the Fourier series of the input. With $\omega_0 = \frac{2\pi}{3}$:

$$\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt \\
&= -\frac{2}{3jk\omega_0}e^{-jk\omega_0 t}\Big|_0^{\frac{1}{2}} \\
&= \frac{2}{3jk\omega_0}[1 - e^{-jk\omega_0/2}]
\end{aligned}$$

Previously, we determined that if the input is $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ and we know $H(\omega)$, then the output is $y(t) = \sum_{k=-\infty}^{\infty} a_k H(k\omega)e^{jk\omega_0 t}$. But take a look at $H(\omega)$— it's nonzero only in two small intervals of ω. So let's take a look at what $\omega = k\omega_0$ can go through this filter. We summarize the results in Figure 3. Since $H(\omega)$ is nonzero only for $k = 3$ and $k = -3$, the filter only passes the third harmonic (yes, don't forget the negative stuff). From our formula for a_k, we find $a_3 = \frac{2}{3j\pi}$. Since $x(t)$ is real, $a_{-3}^* = a_3$.

k	$\omega = k\omega$
0	0
1	$\frac{2\pi}{3}$
2	$\frac{4\pi}{3}$
3	2π
4	$\frac{8\pi}{3}$
5	$\frac{10\pi}{3}$

Figure 3: ω as a function of k.

Our output is then:

$$
\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} a_k H(k\omega) e^{jk\omega_0 t} \\
&= a_3 e^{j3\omega_0 t} + a_{-3} e^{-j3\omega_0 t} \\
&= \frac{4}{3\pi} \sin 2\pi t
\end{aligned}
$$

5 A Sample Midterm Problem

[Fall 1994 midterm] In Figure 4a, $x(t)$ is sketched. Since $x(t)$ is periodic [and satisfies the Dirichlet conditions], we

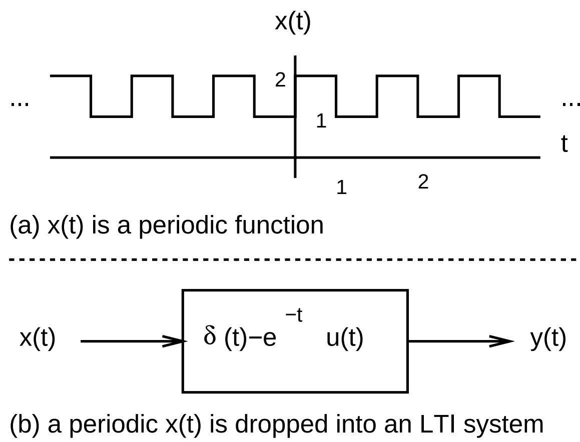

Figure 4: Figures for a midterm problem.

can represent $x(t)$ as a Fourier series: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$. The question asks:

1. what is a_0?
2. what is a_1?
3. what is a_5?

From above, we know that a square wave has FS coefficients $c_k = \frac{1}{k\pi} \sin 2k\pi \frac{T_1}{T}$. We identify $T_1 = \frac{1}{2}$ and $T = 2$, so $c_k = \frac{1}{k\pi} \sin k\pi/2$. But this is not a_k (well, for one, i called it something else). We also need to account for a delay of $\frac{1}{2}$ units and an offset of 1. From our table of interesting FS properties, we find that a delay of t_0 just multiples the FS coefficients by $e^{jk\omega_0 t_0}$. So we need to multiply c_k by $e^{jk\omega_0 \frac{1}{2}} = e^{jk\pi/2} = (-j)^k$; $a_k = (-j)^k c_k$ for all $k \neq 0$. For $k = 0$, this is just the DC level of this signal, which can be found by inspection. $a_0 = \frac{3}{2}$, $a_1 = -\frac{j}{\pi}$, and $a_5 = -\frac{j}{5\pi}$.

Exercise In Figure 4b, a periodic function $x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin k\pi/2}{k\pi} e^{jk\pi t}$ is input to an LTI system with impulse response $h(t) = \delta(t) - E^{-t} u(t)$. The output is $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\pi t}$.

1. what is b_0?
2. what is b_1?
3. what is the power in the fundamental frequency of $x(t)$?
4. what is the power in the fundamental frequency of $y(t)$?

Verify that $H(\omega) = \frac{j\omega}{1+j\omega}$, that $b_0 = 0$, that $b_1 = \frac{j}{1+j\pi}$, that the power in fundamental of $x(t)$ is $\frac{2}{\pi^2}$, and that the power in fundamental of $y(t)$ is $\frac{2}{1+\pi^2}$.

6 A Look Ahead

Fourier series is for periodic signals. Next we'll tackle aperiodic signals with the Fourier transform and try to draw some relationship between the two.