Notes 02 largely plagiarized by %khc

1 System Properties

We'll be using a number of definitions in our discussion of systems. Each of these definitions describes some property of a system. They *do not* refer to properties of signals; it makes no sense for a signal to be linear, time invariant, causal, memoryless, or stable.

Linearity To tell if a system is linear, ask the question:
If $x_1(t) \to y_1(t)$ and $x_2(t) \to y_2(t)$, does $\alpha x_1(t) + \beta x_2(t) \to \alpha y_1(t) + \beta y_2(t)$?

Rephrased:
If i multiply the input by a constant, does the output get multiplied by the same constant?
If my input is the sum of a number of signals $x_i(t)$, is the total output of the system going to be the sum of system responses $y_i(t)$ to each of the individual input signals $x_i(t)$?

The first property is called *homogeneity*. The second property is called *additivity*, or *superposition*. You have already used it in solving circuits by superposition; imagine solving them without linearity and see how much hair loss results.

Both homogeneity and additivity are necessary parts of the definition of linearity.

Exercise Show that $y(t) = \frac{x^2(t)}{x(t-1)}$ satisfies homogeneity, but not additivity. Consequently, the system is not linear.

You have also seen linearity in studying differential equations, equations of the form $\sum_{n=0}^{N} a_n y^{(n)} = \sum_{m=0}^{M} b_m x^{(m)}$, where x is the input to the system, y is the output, and the a_n and b_m are constants. We will return to this later on.

For examples of linear systems, consider all the circuit stuff that you have learned so far. Resistors satisfying Ohm's law, capacitors satisfying $i = C\dot{v}$, and inductors satisfying $v = L\dot{i}$, are all linear elements. In real life™, just about everything is nonlinear though. To make our lives easier, we usually linearize the system about some operating point and then assume the system stays close to that operating point, so that our linearized models hold [such as you did if you have studied small signal analysis of transistors].

Exercise Prove that the systems described by $y(t) = cx(t)$ and $y(t) = cx(t-2)$ are linear and that the one described by $y(t) = cx^2(t-2)$ is not.

Exercise Prove that for a system to be linear, zero input must give zero output [hint: let β equal $-\alpha$].

Time invariance To tell if a system is time invariant, ask the question:
If $x(t) \to y(t)$, does $x(t-t_0) \to y(t-t_0)$?
Note that t_0 can be positive or negative, indicating that the input can be either delayed or advanced.

Time invariance is a useful simplifying assumption. If a system responded to an input depending on exactly when that input happened (as opposed to responding to the same inputs in the same manner regardless of when those inputs occur), then your life would be more complicated.

For a prosaic example, consider calling me up at 11:00AM. i will speak to you nicely. If i were a time invariant system, if you called me up at 3:30AM, i would speak to you nicely as if you called me up at 11:00AM, because i wouldn't care about the time. On the other hand, if i were a time varying system, if you called me up at 3:30AM, i would use call return and flame you repeatedly. And then you would fail.

Exercises For the systems mentioned in the previous exercise for linearity, determine if they are time invariant.

Memorylessness To tell if a system is memoryless, ask the question:
Does the output at any given time depend *only* on the input at that instant in time?

A strange thing is that if the output of a system depends on the input at a future point in time, the system has memory, even though the memory is of the future.

Exercise Determine the values of t_0 for which the system described by $y(t) = x(t-t_0)$ is memoryless.

Causality To tell if a system is causal, ask the question:
Does the output of the system at the current time depend only on the current and past input?
Alternatively, ask the question:
If i keep the inputs to a given system the same up to a certain point in time, do the outputs stay the same up to that same point in time? Alternatively, ask the question:
Is the system nonanticipatory? Does the system wait for my input to change, and then react to that change at that same time?

All real world systems are causal if time is the independent variable. In image processing systems, we usually employ space as the independent variable [think of a picture as lying in the xy plane, and then index the pixels of the picture in x and y], in which case causality is much less of an issue, since we have the whole picture at a given instance in time.

Exercise Determine the range of values for t_0 such that $y(t) = x(t - t_0)$ is causal.

Exercise Is the system $y(t) = x(2t)$ causal? How about $y(t) = x(-2t)$?

Memorylessness implies causality [hint: look at the definitions above]. By contraposition, not causal implies not memoryless. *No other relationships between causality and memorylessness exist;* create them at your peril.

Bounded-input bounded-output stability (BIBO stability) To tell if a system is BIBO stable, ask the question:
If $x(t)$ is bounded [$|x(t)| < M < \infty$ for all t], is $y(t)$ bounded?

If you are faced with determining stability, try the constant and step functions as input and then look at the output. *Do not use the impulse as an input. It is not bounded.*

Stability is always a concern, but it is application dependent. X-ray machines should be stable, cars less so for performance purposes. The oscillator in your watch or computer should be marginally stable, otherwise time-keeping would be painful without periodicity. Bombs should be unstable.

Once again, note that linearity, time invariance, causality, memory, and BIBO stability are system properties. *It makes no sense whatsoever to use these properties to describe inputs, outputs, or other such signals.* [Well, except for causality; DSP guys sometimes refer to signals as being causal. They mean that the signal is zero for time $t < 0$.]

In the future We will have three other methods of determining BIBO stability and one other method of determining causality.

2 Some Examples

Examples are always useful, if nothing better than as fodder for the pattern matcher. In the following discussion, note that the $x(t)$ are the system inputs and the $y(t)$ are the system outputs.

Consider an amplifier system A, which multiplies its input by a constant a. An equation describing its function can be written as $y(t) = A[x(t)] = ax(t)$.

- Is it linear? If $y_1(t) = A[x_1(t)] = ax_1(t)$ and $y_2(t) = A[x_2(t)] = ax_2(t)$, then

$$\begin{aligned} A[\alpha_1 x_1(t) + \alpha_2 x_2(t)] &= a(\alpha_1 x_1(t) + \alpha_2 x_2(t)) \\ &= \alpha_1 a x_1(t) + \alpha_2 a x_2(t) \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

  Yes.

- Is it time invariant? If $y_1(t) = A[x_1(t)] = ax_1(t)$, then

$$\begin{aligned} A[x_1(t - t_0)] &= a x_1(t - t_0) \\ &= y_1(t - t_0) \end{aligned}$$

  Yes.

- Is it memoryless? Yes. The output at any time depends only on the current input.

- Is it causal? Yes. All memoryless systems are causal.

Exercise Consider an ideal wire, which can be described by the function $y(t) = x(t)$. Is it linear? time invariant? causal? memoryless?

Now let's look at an integrator, which can be described by $y(t) = I[x(t)] = \int_{t_0}^{t} x(\tau)d\tau$.

- Is it linear? If $y_1(t) = I[x_1(t)] = \int_{t_0}^{t} x_1(\tau)d\tau$ and $y_2(t) = I[x_2(t)] = \int_{t_0}^{t} x_2(\tau)d\tau$, then

$$\begin{aligned} I[\alpha_1 x_1(t) + \alpha_2 x_2(t)] &= \int_{t_0}^{t} (\alpha_1 x_1(\tau) + \alpha_2 x_2(\tau))d\tau \\ &= \alpha_1 \int_{t_0}^{t} x_1(\tau)d\tau + \alpha_2 \int_{t_0}^{t} x_2(\tau)d\tau \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

  Yes.

- Is it time invariant? If $y_1(t) = I[x_1(t)] = \int_{t_0}^{t} x_1(\tau)d\tau$ then

$$\begin{aligned} I[x_1(t - t_1)] &= \int_{t_0}^{t} x_1(\tau - t_1)d\tau \\ &= \int_{t_0 - t_1}^{t - t_1} x_1(v)dv \\ &\neq y_1(t - t_0) \end{aligned}$$

  No, since the lower limit is not t_0. But if t_0 were $-\infty$, the system would be time invariant.

- Is it memoryless? No. The output at any time depends not only on the current input but on the input previous to the current time.

- Is it causal? Yes. The integral only goes to t and does not include future time.

See Table 1 below for more examples.

	linear	time invariant	causal	memoryless	BIBO stable
$y(t) = 2x(t) + 1$	no	yes	yes	yes	yes
$y(t) = x^2(t+1)$	no	yes	no	no	yes
$\dot{y}(t) - y(t) = x(t)$, initially at rest	yes	yes	yes	no	no
$\dot{y}(t) + y(t) = x(t)$, initially at rest	yes	yes	yes	no	yes
$\dot{y}(t) + y(t) = tx(t)$, initially at rest	yes	no	yes	no	no
$y(t) = \dot{x}(t)$	yes	yes	no	no	no
$y(t) = x(t)u(t)$	yes	no	yes	yes	yes
$y(t) = x(t)\cos\omega_c t$	yes	no	yes	yes	yes
$y(t) = x(t)[\delta(t) + \delta(t+2)]$	yes	no	yes	yes	no
$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)dt$	yes	yes	?	?	?

Table 1: Fun and interesting examples.

3 ZIR and ZSR

Consider the first order system in Figure 1 below.

Writing KCL at the v_o node gives:

$$\frac{v_i - v_o}{R} = C\dot{v}_o$$

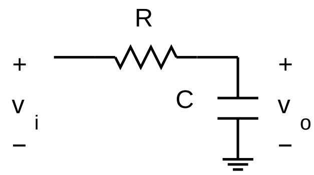

Figure 1: An RC circuit.

What if the voltage across the capacitor was 4V at $t = 0$ and the input was a step? If we went ahead, found the homogeneous and particular solutions, and otherwise cranked through the problem the same way we always did in math classes:

$$\begin{aligned} v_{o,homogenous} &= C_{oh} e^{-t/RC} \\ v_{o,particular} &= 1 \\ v_o &= v_{o,homogenous} + v_{o,particular} \\ &= 1 + C_{oh} e^{-t/RC} \\ &= 1 + 3e^{-t/RC} \end{aligned}$$

Does this make sense? The voltage on the capacitor is 4V initially, and decays to 1V. OK.

But there is another way to look at the problem. Let's first kill the input and just see what happens to the initial condition; this should give us the zero input response (ZIR).

$$\begin{aligned} v_{o,homogenous} &= C_{oh} e^{-t/RC} \\ v_{o,particular} &= 0 \\ v_{o,ZIR} &= v_{o,homogenous} + v_{o,particular} \\ &= C_{oh} e^{-t/RC} \\ &= 4e^{-t/RC} \end{aligned}$$

Sanity check: the 4V on the capacitor decays exponentially.

But there is another way to look at the problem. Now let's zero the initial conditions ($v_o(0) = 0$) and just see what happens with the step input. This should give us the zero state response (ZSR).

$$\begin{aligned} v_{o,homogenous} &= C_{oh} e^{-t/RC} \\ v_{o,particular} &= 1 \\ v_{o,ZSR} &= v_{o,homogenous} + v_{o,particular} \\ &= 1 + C_{oh} e^{-t/RC} \\ &= 1 - e^{-t/RC} \end{aligned}$$

Sanity check: the capacitor voltage rises from 0V to 1V.

Now, i claim that the solution we found using the good old math method is the sum of the ZIR and the ZSR.

$$\begin{aligned} v_o &= v_{o,ZIR} + v_{o,ZSR} \\ &= 1 + 3e^{-t/RC} \end{aligned}$$

We are allowed to do this because superposition applies.

So if we use the good old math method, we find that particular solution of the math method contributes to the ZSR only, and that the homogeneous solution shows up in both the ZIR and the ZSR. Why does part of the homogeneous solution show up in the ZSR? Well, the output resists changing quickly because of the capacitor. In other words, we can think of the input as exciting the dynamics of the system, as represented by the decaying exponential.

For fun For those of you who know the Laplace transform, solve the equation using that technique and try to identify the ZIR and the ZSR.

4 More on the ZIR and the ZSR

Let's consider the second order system in Figure 2 below.

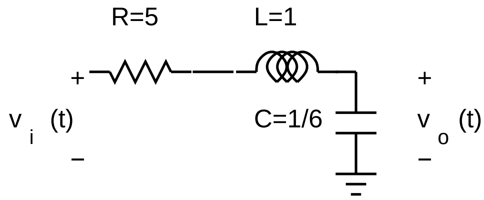

Figure 2: An RLC circuit.

Writing the loop equations for this circuit gives:

$$\begin{aligned} \frac{v_i - (v_L + v_o)}{R} &= i_L \\ i_L &= C\dot{v}_o \\ v_L &= L\dot{i}_L \end{aligned}$$

where i_L is the current flowing into the inductor and v_L is the voltage across the inductor.

Solving these equations gives:

$$\begin{aligned} \frac{1}{LC}v_i &= \ddot{v}_o + \frac{R}{L}\dot{v}_o + \frac{1}{LC}v_o \\ 6v_i &= \ddot{v}_o + 5\dot{v}_o + 6v_o \end{aligned}$$

Zeroing the input and solving, we get the homogeneous solution:

$$v_{oh}(t) = C_1 e^{-2t} + C_2 e^{-3t}$$

The particular solution depends on what $v_i(t)$ is. Let's use $v_i(t) = \cos t$. We then use the method of undetermined coefficients (look at input, guess the form of the output with unknown constants in it, substitute the guess into the original LDE, solve for those constants) to find the particular solution.

$$\begin{aligned} v_{op}(t) &= A\cos t + B\sin t \\ 6v_i &= \ddot{v}_{op} + 5\dot{v}_{op} + 6v_{op} \\ v_{op}(t) &= \frac{3}{5}\cos t + \frac{3}{5}\sin t \end{aligned}$$

From the initial conditions, we can find C_1 and C_2 in the homogeneous solution. Let's use $v_o(0) = 2$ and $i_L(0) = 3$. Since $i_L = C\dot{v}_o$, we have $\dot{v}_o = 18$. Grinding through the algebra:

$$\begin{aligned} v_o(t) &= C_1 e^{-2t} + C_2 e^{-3t} + \frac{3}{5}\cos t + \frac{3}{5}\sin t \\ v_o(0) &= C_1 + C_2 + \frac{3}{5} \\ \dot{v}_o(0) &= -2C_1 - 3C_2 + \frac{3}{5} \\ v_o(t) &= \frac{108}{5}e^{-2t} - \frac{101}{5}e^{-3t} + \frac{3}{5}\cos t + \frac{3}{5}\sin t \end{aligned}$$

Why do we bother? Physical systems can be modeled by systems of differential equations. In techniques which we will not study in this class (but you can ask me about them anyway), by appropriately linearizing about certain operating points, we can reduce tons of gory nonlinear terms into simpler LDEs for which some hopefully useful analysis can be performed.

Let's find the ZIR. Let's use $v_o(0) = 2$ and $\dot{v}_o = 18$ as above. Start from the homogeneous solution, since the particular solution is identically zero; this corresponds to killing the input.

$$\begin{aligned}
v_{o,\text{ZIR}}(t) &= C_1 e^{-2t} + C_2 e^{-3t} \\
v_{o,\text{ZIR}}(0) &= C_1 + C_2 \\
\dot{v}_{o,\text{ZIR}}(0) &= -2C_1 + -3C_2 \\
v_{o,\text{ZIR}}(t) &= 24e^{-2t} + -22e^{-3t}
\end{aligned}$$

Now for the ZSR. We zero the initial conditions, so $v_o(0) = 0$ and $\dot{v}_o = 0$. But we still have to find the response of the system to being driven by the input. We have most of it already in the sum of the particular and the homogeneous solutions. Using the zeroed initial conditions, we get:

$$\begin{aligned}
v_{o,\text{ZSR}}(t) &= C_1 e^{-2t} + C_2 e^{-3t} + \frac{3}{5}\cos t + \frac{3}{5}\sin t \\
v_{o,\text{ZSR}}(0) &= C_1 + C_2 + \frac{3}{5} \\
\dot{v}_{o,\text{ZSR}}(0) &= -2C_1 - 3C_2 + \frac{3}{5} \\
v_{o,\text{ZSR}}(t) &= -\frac{12}{5}e^{-2t} + \frac{9}{5}e^{-3t} + \frac{3}{5}\cos t + \frac{3}{5}\sin t
\end{aligned}$$

As a check, we can add the ZIR and the ZSR together to get the same thing we got from cranking through the math in the previous section.

The ZIR is the response of the system to the initial conditions only. For the given set of initial conditions $v_o(0) = 2$ and $\dot{v}_o = 18$, the system will always respond in the fashion $v_{o,\text{ZIR}}(t) = 24e^{-2t} + -22e^{-3t}$. The two decaying exponentials result from the dynamics of the system.

The ZSR is the response of the system to the input only. For the given input $\cos t$, the system will always respond in the fashion $v_{o,\text{ZSR}}(t) = -\frac{12}{5}e^{-2t} + \frac{9}{5}e^{-3t} + \frac{3}{5}\cos t + \frac{3}{5}\sin t$. Note that the two decaying exponentials; the input excites the system dynamics, as well as showing up in the output as both a cosine and a sine.

Thus, it is not useful to say that the particular solution corresponds to the ZSR exactly and that the homogeneous solution is the ZIR. Parts of the homogeneous solution show up in both the ZIR and the ZSR. However, the entirety of the particular solution does show up as part of the ZSR.

Exercise Go back through this section and the previous one. Check the algebra, see where linearity shows up, and understand the definitions of the ZIR and the ZSR. If you do not know how to solve an LDE, read the review modules, since this is something you should already know.

5 A Look Ahead

Please review the various system definitions. In particular, we will need the definitions for linearity and time invariance. With the sifting integral, we will then derive the convolution integral.

We will also have a much better way (read: simpler) of solving LDEs after we get to the Fourier and Laplace transforms.