EECS 120 Fall 2002 Homework 12 Solutions

1. HV 3.1(h) $x[n] = (\frac{1}{2})^{|n|}$

   As shown in class:

$$\alpha^{|n|} \overset{z}{\longleftrightarrow} \frac{(\alpha - \alpha^{-1})z^{-1}}{(1-\alpha z^{-1})(1-\alpha^{-1}z^{-1})} = \frac{(\alpha - \alpha^{-1})z}{(z-\alpha)(z-\alpha^{-1})}$$

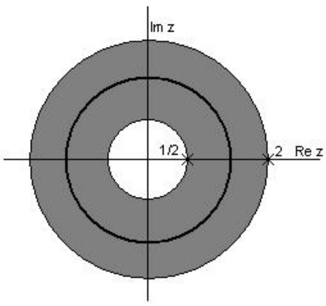
$$ROC = \{z \mid |z| > |\alpha|\} \cap \{z \mid |z| < |\alpha^{-1}|\}$$

   With $\alpha = \frac{1}{2}$:

$$(\frac{1}{2})^{|n|} \overset{z}{\longleftrightarrow} \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{-\frac{3}{2}z}{(z-\frac{1}{2})(z-2)}$$

   X(z) zero at $z = 0$. X(z) poles at $z = \frac{1}{2}$, $z = 2$

$$ROC = \{z|\frac{1}{2} < |z| < 2\}$$



   HV 7.1(j) $x[n] = (\frac{1}{2})^n u[n] + (\frac{1}{3})^n u[n-1]$

$$X(z) = \sum_{n=0}^{\infty}(\frac{1}{2}z^{-1})^n + \sum_{n=-\infty}^{-1}(\frac{1}{3}z^{-1})^n = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{3}z^{-1}}$$

$$|z| > \frac{1}{2} \cap |z| < \frac{1}{3}$$

   DOESN'T EXIST

2. HV 7.5(b) $x[n] = n(((\frac{1}{2})^n u[n] * ((\frac{1}{2})^n u[n])$

$$X(z) = -z\frac{d}{dz}(\frac{1}{(1-\frac{1}{2}z^{-1})^2}) = -z\frac{(-2)(\frac{1}{2}z^{-2})}{(1-\frac{1}{2}z^{-1})^3} = \frac{z^{-1}}{(1-\frac{1}{2}z^{-1})^3}\ ,\ |z| > \frac{1}{2}$$

3. HV 7.6 (a) $y[n] = x[n-4]$

$$Y(z) = \frac{z}{z^2+4}z^{-4}$$

   HV 7.6 (f) $y[n] = \underbrace{x[n] * x[n] * \cdots * x[n] * x[n]}_{m \text{ times}}$

$$Y(z) = (\frac{z}{z^2+4})^m$$

4. HV 7.8 (a) $X(z) = \frac{\frac{1}{4}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$, $|z| > \frac{1}{2} \longrightarrow$ *casual*

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1-\frac{1}{4}z^{-1}}$$

$$x[n] = ((\frac{1}{2})^n - (\frac{1}{4})^n)u[n]$$

HV 7.8 (b) same as (a) except $|z| < \frac{1}{4} \longrightarrow$ *anticasual*

$$x[n] = ((\frac{1}{4})^n - (\frac{1}{2})^n)u[-n-1]$$

HV 7.8 (c) same as (a) except $\frac{1}{4} < |z| < \frac{1}{2} \longrightarrow$ *two sided*

$$x[n] = -(\frac{1}{4})^n u[n] - (\frac{1}{2})^n u[-n-1]$$

5. HV 7.10 (a)

$$H(z) = \frac{3z^{-1}}{(1-2z^{-1})^2}$$

(i) stable, ROC: $|z| < 2$ (includes $|z| = 1$) $\longrightarrow$ *anticausal*

$$h[n] = -\frac{3}{2}(2)^n nu[-n-1]$$

(ii) causal, ROC: $|z| < 2$

$$h[n] = \frac{3}{2}(2)^n nu[n]$$

6. HV 7.14 (b)

$$\begin{aligned} y[n] &= x[n] - x[n-2] + x[n-4] - x[n-6] \\ Y(z) &= (1 - z^{-2} + z^{-4} - z^{-6})X(z) \\ H(z) &= 1 - z^{-2} + z^{-4} - z^{-6} \\ h[n] &= \delta[n] - \delta[n-2] + \delta[n-4] - \delta[n-6] \end{aligned}$$

Since all poles of $H(z)$ are at $z = 0$, ROC includes $|z| = 1$, and the system is stable.

7. HV 7.18 (e)

$$y[n] - \frac{1}{4}y[n-2] = 6x[n] - 7x[n-1] + 3x[n-2]$$

$$(1 - \frac{1}{4}z^{-2})Y(z) = (6 - 7z^{-1} + 3z^{-2})X(z)$$

$$H^{-1}(z) = \frac{X(z)}{Y(z)} = \frac{1-\frac{1}{4}z^{-2}}{6-7z^{-1}+3z^{-2}} = \frac{(z-\frac{1}{2})(z+\frac{1}{2})}{6z^2-7z+3}$$

Pole: $6z^2 - 7z + 3 = 0$, $z_p = \frac{7 \pm j\sqrt{23}}{12}$. Since $|z_p| = \frac{1}{\sqrt{2}} < 1$, it can be both causal and stable.

8. HV 7.27 (b)

$$y[n] - \frac{1}{9}y[n-2] = x[n-1],\ y[-1] = 0,\ y[-2] = 1,\ x[n] = 3u[n]$$

$$Y(z) - \frac{1}{9}(Y(z)z^{-2} + y[-1]z^{-1} + y[-2]) = X(z)z^{-1} + x[-1], \text{ and } x[-1] = 0$$

$$Y(z)(1 - \frac{1}{9}z^{-2}) = X(z)z^{-1} + \frac{1}{9}(y[-1]z^{-1} + y[-2])$$

$$
\begin{aligned}
Y(z) &= \underbrace{X(z)\frac{z^{-1}}{(1-\frac{1}{9}z^{-2})}}_{\text{forced response}} + \underbrace{\frac{\frac{1}{9}}{1-\frac{1}{9}z^{-2}}}_{\text{natural response}} \\
X(z) &= \frac{3}{(1-z^{-1})} \\
Y(z) &= \frac{3z^{-1}}{(1-\frac{1}{9}z^{-2})(1-z^{-1})} + \frac{\frac{1}{9}}{1-\frac{1}{9}z^{-2}} \\
&= \underbrace{\frac{A_1}{(1-\frac{1}{3}z^{-1})} + \frac{A_2}{(1+\frac{1}{3}z^{-1})} + \frac{A_3}{(1-z^{-1})}}_{\text{forced}} + \underbrace{\frac{B_1}{(1-\frac{1}{3}z^{-1})} + \frac{B_2}{(1+\frac{1}{3}z^{-1})}}_{\text{natural}} \\
&= \frac{-\frac{9}{4}}{(1-\frac{1}{3}z^{-1})} + \frac{-\frac{9}{8}}{(1+\frac{1}{3}z^{-1})} + \frac{\frac{27}{8}}{(1-z^{-1})} + \frac{\frac{1}{18}}{(1-\frac{1}{3}z^{-1})} + \frac{\frac{1}{18}}{(1+\frac{1}{3}z^{-1})} \\
y^f[n] &= -\frac{9}{4}(\frac{1}{3})^n u[n] - \frac{9}{8}(-\frac{1}{3})^n u[n] + \frac{27}{8}u[n] \\
y^n[n] &= -\frac{1}{18}(\frac{1}{3})^n u[n] - \frac{1}{18}(-\frac{1}{3})^n u[n] \\
y[n] &= -\frac{79}{36}(\frac{1}{3})^n u[n] - \frac{77}{72}(-\frac{1}{3})^n u[n] + \frac{27}{8}u[n]
\end{aligned}
$$

9. We begin by Z transofrming the given equation and applying the property we derived in 2(b).

$$
\begin{aligned}
f[n+2] &= f[n+1] + f[n] \\
z^2F(z) - z^2f[0] - zf[1] &= zF(z) - zf[0] + F(z)
\end{aligned}
$$

Substituting the initial conditions, we obtain:

$$
\begin{aligned}
z^2F(z) - z &= zF(z) + F(z) \\
(z^2 - z - 1)F(z) &= z \\
F(z) &= \frac{z}{z^2-z-1} \\
&= \frac{A}{z-\frac{1+\sqrt{5}}{2}} + \frac{B}{z-\frac{1-\sqrt{5}}{2}} \\
A &= z - \frac{1+\sqrt{5}}{2}\frac{z}{z^2-z-1}|_{z=\frac{1+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2\sqrt{5}} \\
B &= z - \frac{1-\sqrt{5}}{2}\frac{z}{z^2-z-1}|_{z=\frac{1-\sqrt{5}}{2}} = -\frac{1-\sqrt{5}}{2\sqrt{5}} \\
F(z) &= \frac{1+\sqrt{5}}{2\sqrt{5}}\frac{1}{z-\frac{1+\sqrt{5}}{2}} - \frac{1-\sqrt{5}}{2\sqrt{5}}\frac{1}{z-\frac{1-\sqrt{5}}{2}} \\
F(z) &= \frac{1+\sqrt{5}}{2\sqrt{5}}\frac{z^{-1}}{1-\frac{1+\sqrt{5}}{2}z^{-1}} - \frac{1-\sqrt{5}}{2\sqrt{5}}\frac{z^{-1}}{z-\frac{1-\sqrt{5}}{2}z^{-1}} \\
f[n] &= \frac{1+\sqrt{5}}{2\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n-1}u[n-1] - \frac{1-\sqrt{5}}{2\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n-1}u[n-1] \\
&= \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n u[n-1] - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n u[n-1]
\end{aligned}
$$

Solving for $n = 33$ in MATLAB, we obtain $f[33] = 3524578$.

10. (a) You might find it easiest to find this by the process of elimination; i.e. by solving (b) $\rightarrow$ (e) first.

This is an allpass filter. It has one pole and one zero. If the pole appears at $z = d_1 = r_1 e^{j\Omega_1}$, the zero should appear at $z = \frac{1}{d_1^*} = \frac{1}{r_1} e^{j\Omega_1}$; i.e., we say that the pole and zero are at "conjugate-reciprocal locations". Here, d_1 is real, $d_1 \approx 0.4$, $\frac{1}{d_1^*} \approx 2.5$

For an allpass filter, $H(z)$ has the form:

$$H(z) = \frac{z - \frac{1}{d_1^*}}{z - d_1}$$

The magnitude squared of the frequency response is:

$$|H(e^{j\Omega})|^2 = (\frac{e^{j\Omega} - \frac{1}{d_1^*}}{e^{j\Omega} - d_1})(\frac{e^{-j\Omega} - \frac{1}{d_1}}{e^{-j\Omega} - d_1^*}) = \frac{1 - \frac{1}{d_1^*} e^{-j\Omega} - \frac{1}{d_1} e^{j\Omega} + \frac{1}{|d_1|^2}}{1 - d_1 e^{-j\Omega} - d_1 e^{j\Omega} + |d_1|^2}$$

Multiply through by:

$$\frac{|d_1|^2}{|d_1|^2} = \frac{d_1 d_1^*}{d_1 d_1^*}$$

$$|H(e^{j\Omega})|^2 = (\frac{1}{|d_1|^2})(\frac{|d_1|^2 - d_1 e^{-j\Omega} - d_1^* e^{j\Omega} + 1}{1 - d_1 e^{-j\Omega} - d_1^* e^{j\Omega} + |d_1|^2}) = \frac{1}{|d_1|^2}$$

which is a constant, independent of Ω. Hence, it matches # **4**.

(b) When $\Omega \to 70^o \simeq 0.4\pi\ rad$, z lies very close to one of the two poles and $|H(e^{j\Omega})|$ gets very large; i.e. $H(e^{j\Omega})$ exhibits strong peaking. Matches # **1**.

(c) When $\Omega \to 0$, z lies closest to the two poles, and $|H(e^{j\Omega})|$ takes on its largest value. Matches # **3**.

(d) When $\Omega \to 0$, z lies closest to the two zeros, and $|H(e^{j\Omega})|$ takes on its smallest value. Matches # **5**.

(e) When $\Omega = 0$, z coincides with a zero of $H(z)$, and $H(e^{j\Omega}) = 0$. Matches # **2**.