CS 70 FALL 2006 — DISCUSSION #10

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1. Administrivia

(1) Course Information
- Midterm #2 will be held in-class Friday
- No homework due this week

2. Life Insurance

As an extended example of probability, we analyze a simple life insurance system. A real system would be too cumbersome to look at, so we make many simplifications here.

Here are the basic rules for our system:

1. You pay b dollars to the insurance company when you are born. You never have to pay again.
2. If you die before age c, the company pays your beneficiaries d dollars.
3. The insurance company is non-profit, so just wants to break even.

Given these rules, what should the insurance company set as the values of b and d, in terms of c? Let X be the age at which a person dies. The fraction of its customers the insurance pays is then the fraction of those that die before age c, or $\Pr(X < c)$. Then b and d are related by $b = d \cdot \Pr(X < c)$.

Lets do a detailed example, where $c = 60$ and $d = \$1,000,000$. We need to compute $\Pr(X < 60)$.

2.1. Distribution of Death.

Before we can calculate $\Pr(X < 60)$, we need to know what the distribution of X looks like. First, lets assume that nobody lives past 100. Now we cant just take the distribution to be uniform in the range $\{1, \ldots, 100\}$, since a person is more likely to die as they get older. So let's assume a linear distribution of the form $\Pr(X = k) = \frac{k}{N}$ for $k \in \{1, \ldots, 100\}$.

Exercise 1. Calculate the constant N in order to ensure the probabilities sum to 1.

2.2. Life Expectancy.

The first thing we should calculate is the expected age at which a person dies.

Exercise 2. Calculate $\mathbb{E}[X]$. Use the identity $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

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Knowing just the expectation is not enough to calculate $\Pr(X < 60)$. Consider the two distributions (A) where $\Pr(X = 67) = 1$ and (B) where $\Pr(X = 55) = \Pr(X = 79) = 0.5$. In (A), $\Pr(X < 60) = 0$, whereas in (B) $\Pr(X < 60) = 0.5$. Notice that in both cases $\mathbb{E}[X] = 67$.

The variance is what makes the difference in the above distributions. It is variance that makes insurance useful. If there were no variance, everyone would know when they would die and thus no one would need or provide life insurance.

2.3. **Variance and Chebyshevs Inequality.** We proceed by calculating the variance of the age at which a person dies.

Exercise 3. Calculate $\text{Var}(X)$, using the identity $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$.

Now recall Chebyshevs inequality

$$\Pr(|X - \mathbb{E}[X]| > r) \quad \leq \quad \frac{\text{Var}(X)}{r^2} \quad .$$

Exercise 4. Now use Chebyshev's inequality to upper-bound $\Pr(X < 60)$. What is the problem with this bound?

Exercise 5. When does Chebyshev's give a bound less than 1?

Even in general, Chebyshevs still gives us a weak bound. Its usefulness is due to the fact that it is easy to compute and only requires knowledge of the expectation and variance of a random variable.

2.4. **Exact Solution.** In this case, since the distribution is so simple, we can compute $\Pr(X < 60)$ directly.

Exercise 6. Compute this probability directly, and thus determine the appropriate b that the insurance company should charge clients at birth.