Problem Set 10

1. **Independence**

   (a) Show that, for independent random variables X, Y, we have $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$ (Hint: Show first that, even if the r.v.'s are *not* independent, $\mathbf{E}[XY] = \sum_a \sum_b ab \cdot \Pr[X = a \wedge Y = b]$.)

   (b) Give a simple example to show that the conclusion of part (a) is not necessarily true when X and Y are not independent.

2. **Pairwise Independence**

   This problem illustrates that events may be pairwise independent but not mutually independent. Two fair dice are thrown. Consider the following three events: $A =$ the first die is odd; $B =$ the second die is odd; $C =$ the sum of the dice is odd.

   (a) Show by directly counting sample points that events A, B are independent; that B, C are independent; and that A, C are independent.

   (b) Show that the three events are **not** mutually independent.

3. **Variance**

   (a) Let X and Y be independent random variables. Express $Var(X - Y)$ in terms of $Var(X)$ and $Var(Y)$.

   (b) Given a random variable X with $E(X) = \mu$ and $Var(X) = \sigma^2$, let the random variable $Y = XZ$ where Z is defined as follows: Flip a fair coin. If it comes up H, then $Z = 1$, otherwise $Z = -1$. Find $Var(Y)$.

4. **Chebyshev**

   Each pixel in a 32×8 display is turned on or off with equal probability. The display shows a horizontal line if all 8 pixels in a given row are turned on. Let X denote the number of horizontal lines that the display shows.

   - What is $E[X]$?
   - What is $Var[X]$?
   - how that $Pr[X \geq 2] \leq 1/25$.

5. **Runs**

   A biased coin is tossed n times, and heads shows with probability p on each toss. A *run* is a maximal sequence of throws which result in the same outcome, so that, for example, the sequence $HHTHTTH$ contains five runs.

   - Show that the expected number of runs is $1 + 2(n - 1)p(1 - p)$.

- What is the variance of the number of runs?

6. **Extra Credit**
   I think of two distinct real numbers between 0 and 1 but do not reveal them to you. I now choose one of the two numbers at random and give it to you. Can you give a procedure for guessing whether you were shown the smaller or the larger of the two numbers, such that your guess is correct with probability *strictly* greater than 0.5 (although exactly how much better than 0.5 may depend on the actual values of the two numbers?