CS61B Lecture #23

Today: Trees

Readings for Today: *Data Structures*, Chapter 5

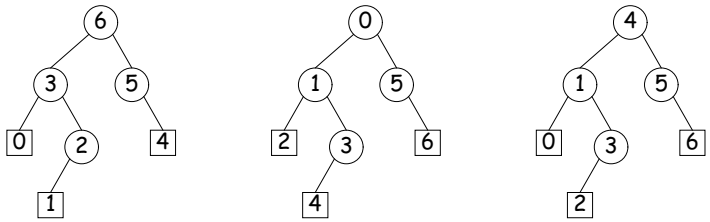
Readings for Next Topic: *Data Structures*, Chapter 6

A Recursive Structure

- Trees naturally represent recursively defined, hierarchical objects with more than one recursive subpart for each instance.
- Common examples: expressions, sentences.
  - Expressions have definitions such as "an expression consists of a literal or two expressions separated by an operator."
- Also describe structures in which we recursively divide a set into multiple subsets.

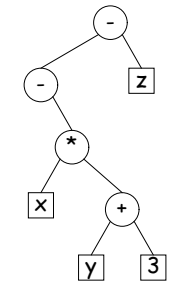
Fundamental Operation: Traversal

- *Traversing a tree* means enumerating (some subset of) its nodes.
- Typically done recursively, because that is natural description.
- As nodes are enumerated, we say they are *visited*.
- Three basic orders for enumeration (+ variations):
  - **Preorder:** visit node, traverse its children.
  - **Postorder:** traverse children, visit node.
  - **Inorder:** traverse first child, visit node, traverse second child (binary trees only).



Postorder *Preorder* *inorder*

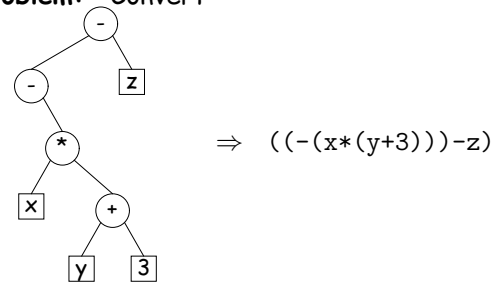
Preorder Traversal and Prefix Expressions

Problem: Convert  ⇒ `(- (- (* x (+ y 3))) z)`


```
static String toLisp (Tree<String> T) {
  if (T == null)
    return "";
  else if (T.degree () == 0)
    return T.label ();
  else {
    String R;  R = "";
    for (int i = 0; i < T.numChildren (); i += 1)
      R += " " + toLisp (T.child (i));
    return String.format ("(%s%s)", T.label (), R);
  }
}
```

Inorder Traversal and Infix Expressions

Problem: Convert



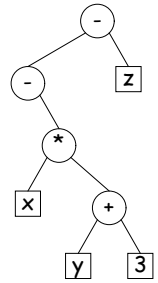
$\Rightarrow$ `((-(x*(y+3)))-z)`

To think about: how to get rid of all those parentheses.

```
static String toInfix (Tree<String> T) {
  if (T == null)
    return "";
  if (T.degree () == 0)
    return T.label ();
  else {
    return String.format ("(%s%s%s)",
                          toInfix (T.left ()), T.label (), toInfix (T.right ())
  }
}
```


Postorder Traversal and Postfix Expressions

Problem: Convert



$\Rightarrow$ `x y 3 +:2 *:2 -:1 z -:2`

```
static String toPolish (Tree<String> T) {
  if (T == null)
    return "";
  else {
    String R;  R = "";
    for (int i = 0; i < T.numChildren (); i += 1)
      R += toPolish (T.child (i)) + " ";
    return String.format ("%s%s:%d", R, T.label (), T.degree ());
  }
}
```


A General Traversal: The Visitor Pattern

```
void preorderTraverse (Tree<Label> T, Action<Label> whatToDo)
{
  if (T != null) {
    whatToDo.action (T);
    for (int i = 0; i < T.numChildren (); i += 1)
      preorderTraverse (T.child (i), whatToDo);
  }
}
```

- What is `Action`?

```
interface Action<Label> {
  void action (Tree<Label> T);
}
```

```
class Print implements Action<String> {
  void action (Tree<String> T) {
    System.out.print (T.label ());
  }
}
```

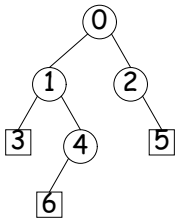
```
preorderTraverse (myTree,
                  new Print ());
```


Times

- The traversal algorithms have roughly the form of the `boom` example in §1.3.3 of *Data Structures*—an exponential algorithm.
- *However,* the role of M in that algorithm is played by the *height* of the tree, not the number of nodes.
- In fact, easy to see that tree traversal is *linear:* $\Theta(N)$, where N is the # of nodes: Form of the algorithm implies that there is one visit at the root, and then one visit for every *edge* in the tree. Since every node but the root has exactly one parent, and the root has none, must be $N-1$ edges in any non-empty tree.
- In positional tree, is also one recursive call for each empty tree, but # of empty trees can be no greater than kN, where k is arity.
- For k-ary tree (max # children is k), $h+1 \le N \le \frac{k^{h+1}-1}{k-1}$, where h is height.
- So $h \in \Omega(\log_k N) = \Omega(\lg N)$ and $h \in O(N)$.
- Many tree algorithms look at one child only. For them, time is proportional to the *height* of the tree, and this is $\Theta(\lg N)$, assuming that tree is *bushy*—each level has about as many nodes as possible.

Level-Order (Breadth-First) Traversal

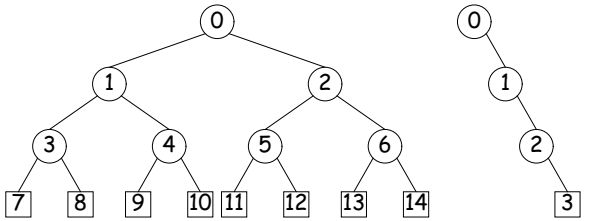
Problem: Traverse all nodes at depth 0, then depth 1, etc:



- One technique: *Iterative Deepening*. For each level, k, from 0 to h, call `doLevel(T,k)`

```
void doLevel (Tree T, int lev) {
  if (lev == 0)
    visit T
  else
    for each non-null child, C, of T {
      doLevel (C, lev-1);
    }
}
```


Iterative Deepening Time?



- Let h be height, N be # of nodes.
- Count # edges traversed (i.e, # of calls, not counting null nodes).
- First (full) tree: 1 for level 0, 3 for level 1, 7 for level 2, 15 for level 3.
- Or in general $(2^1 - 1) + (2^2 - 1) + \ldots + (2^{h+1} - 1) = 2^{h+2} - h \in \Theta(N)$, since $N = 2^{h+1} - 1$ for this tree.
- Second (*right leaning*) tree: 1 for level 0, 2 for level 2, 3 for level 3.
- Or in general $(h+1)(h+2)/2 = N(N+1)/2 \in \Theta(N^2)$, since $N = h+1$ for this kind of tree.

Iterative Traversals

- Tree recursion conceals data: a *stack* of nodes (all the `T` arguments) and a little extra information. Can make the data explicit, e.g.:

```
void preorderTraverse2 (Tree T<T>, Action whatToDo) {
  Stack s = new Stack ();
  s.push (T);
  while (! s.isEmpty ()) {
    Tree node = (Tree) s.pop ();
    if (node == null)
      continue;
    whatToDo.action (node);
    for (int i = node.numChildren ()-1; i >= 0; i -= 1)
      s.push (node.child (i));
  }
}
```

- To do a breadth-first traversal, use a queue instead of a stack, replace `push` with `add`, and `pop` with `removeFirst`.
- Makes breadth-first traversal worst-case linear time in all cases, but also linear *space* for "bushy" trees.

Iterators for Trees

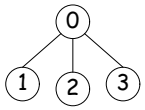
- Frankly, iterators are not terribly convenient on trees.
- But can use ideas from iterative methods.

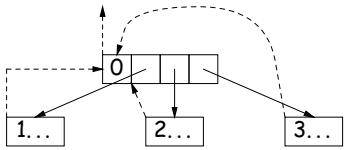
```
class PreorderTreeIterator<T> implements Iterator<T> {
  private Stack<Tree<T>> s = new Stack<Tree<T>> ();

  public PreorderTreeIterator (Tree<T> T) { s.push (T);  }

  public boolean hasNext () { return ! s.isEmpty (); }
  public T next () {
    Tree<T> result = s.pop ();
    for (int i = result.numChildren ()-1; i >= 0; i -= 1)
      s.push (result.child (i));
    return result.label ();
  }
  void remove () { throw new UnsupportedOperationException (); }
}
```

Example: (what do I have to add to class `Tree` first?)

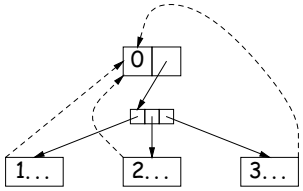
```
for (String label : aTree) System.out.print (label + " ");
```

Tree Representation

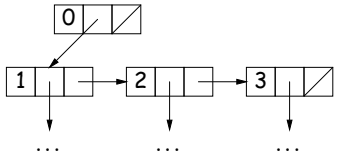




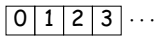
(a) Embedded child pointers
(+ optional parent pointers)



(b) Array of child pointers
(+ optional parent pointers)



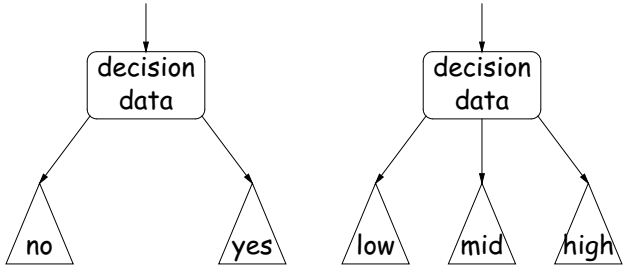
(c) child/sibling pointers



(d) pre-order array
(complete trees)

Divide and Conquer

- Much (most?) computation is devoted to finding things in response to various forms of query.
- Linear search for response can be expensive, especially when data set is too large for primary memory.
- Preferable to have criteria for *dividing* data to be searched into pieces recursively
- Remember figure for $\lg N$ algorithms: at 1μsec per comparison, could process 10^{300000} items in 1 sec.
- Tree is a natural framework for the representation:



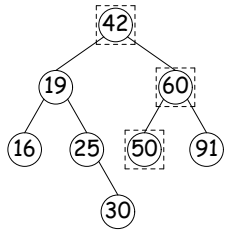
Binary Search Trees

Binary Search Property:

- Tree nodes contain *keys*, and possibly other data.
- All nodes in left subtree of node have *smaller* keys.
- All nodes in right subtree of node have *larger* keys.
- "Smaller" means any complete transitive, anti-symmetric ordering on keys:
  - exactly one of $x \prec y$ and $y \prec x$ true.
  - $x \prec y$ and $y \prec z$ imply $x \prec z$.
  - (To simplify, won't allow duplicate keys this semester).
- E.g., in dictionary database, node label would be (*word, definition*): *word* is the key.

Finding

- Searching for 50 and 49:

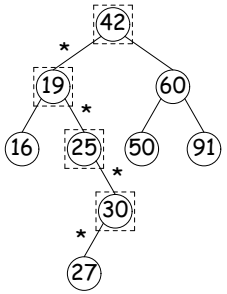



```
/** Node in T containing L,
 *  or null if none */
static BST find(BST T, Key L) {
  if (T == null)
    return T;
  if (L.keyequals (T.label()))
    return T;
  else if (L ≺ T.label())
    return find(T.left(), L);
  else
    return find(T.right (), L);
}
```

- Dashed boxes show which node labels we look at.
- Number looked at proportional to height of tree.

Inserting

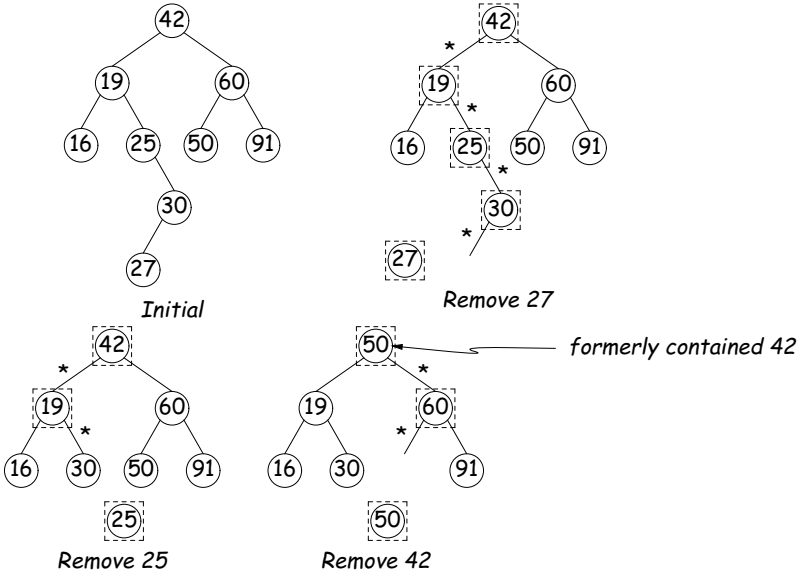
- Inserting 27




```
/** Insert L in T, replacing existing
 *  value if present, and returning
 *  new tree. */
BST insert(BST T, Key L) {
  if (T == null)
    return new BST(L);
  if (L.keyequals (T.label()))
    T.setLabel (L);
  else if (L ≺ T.label())
    T.setLeft(insert (T.left (), L));
  else
    T.setRight(insert (T.right (), L));
  return T;
}
```

- Starred edges are set (to themselves, unless initially null).
- Again, time proportional to height.

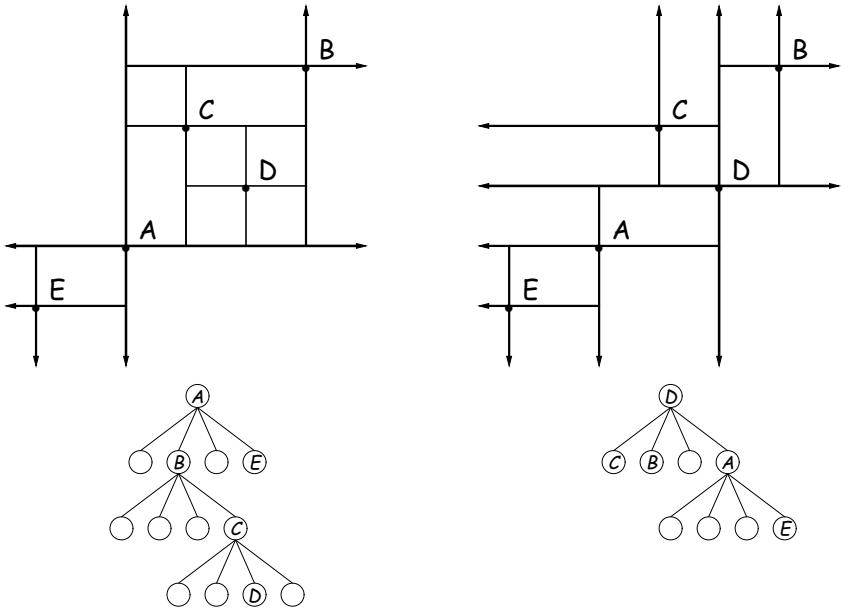
Deletion



A Leap Ahead: Quadtrees

- Want to *index* information about locations so that items can be retrieved by position.
- Quadtrees do so using standard data-structuring trick: *Divide and Conquer*.
- Idea: divide (2D) space into four *quadrants*, and store items in the appropriate quadrant. Repeat this recursively with each quadrant that contains more than one item.
- Original definition: a quadtree is either
  - Empty, or
  - An item at some position (x, y), called the root, plus
  - four quadtrees, each containing only items that are northwest, northeast, southwest, and southeast of (x, y).
- Big idea is that if you are looking for point (x', y') and the root is not the point you are looking for, you can narrow down which of the four subtrees of the root to look in by comparing coordinates (x, y) with (x', y').

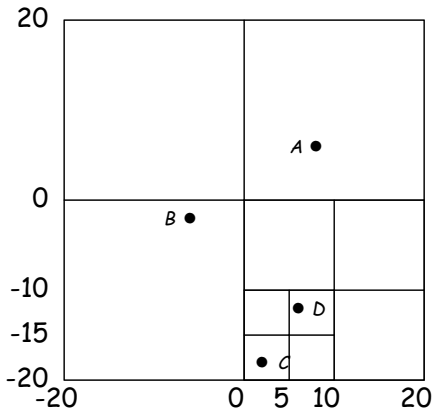
Classical Quadtree: Example



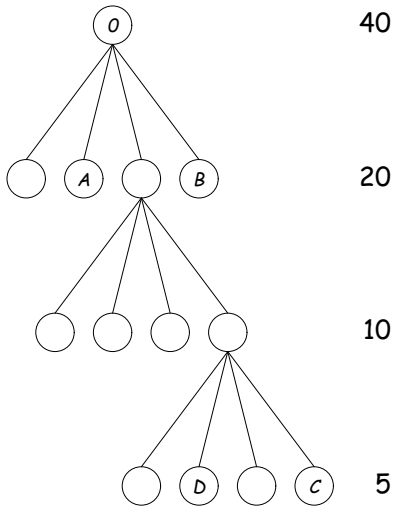
Point-region (PR) Quadtrees

- For our project, it is good to be able to *delete* items from a tree: when a particle moves, the subtree that it goes in may change.
- Difficult to do with the classical data structure above, so we'll define instead:
- A quadtree consists of a bounding rectangle, B and either
  - Zero up to a small number of items that lie in that rectangle, or
  - Four quadtrees whose bounding rectangles are the four quadrants of B (all of equal size).
- A completely empty quadtree can have an arbitrary bounding rectangle, or you can wait for the first point to be inserted.

Example of PR Quadtree



(≤ 1 point per leaf)



Navigating PR Quadtrees

- To find an item at (x, y) in quadtree T,
  1. If (x, y) is outside the bounding rectangle of T, or T is empty, then (x, y) is not in T.
  2. Otherwise, if T contains a small set of items, then (x, y) is in T iff it is among these items.
  3. Otherwise, T consists of four quadtrees. Recursively look for (x, y) in each (however, step #1 above will cause all but one of these bounding boxes to reject the point immediately).
- Similar procedure works when looking for all items within some rectangle, R:
  1. If R does not intersect the bounding rectangle of T, or T is empty, then there are no items in R.
  2. Otherwise, if T contains a set of items, return those that are in R, if any.
  3. Otherwise, T consists of four quadtrees. Recursively look for points in R in each one of them.

Insertion into PR Quadtrees

Various cases for inserting a new point N, showing initial state $\Longrightarrow$ final state.