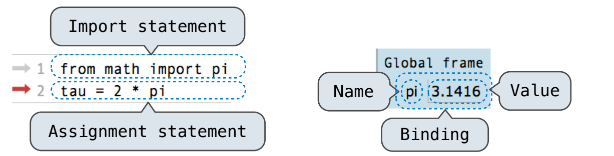


Code (left):

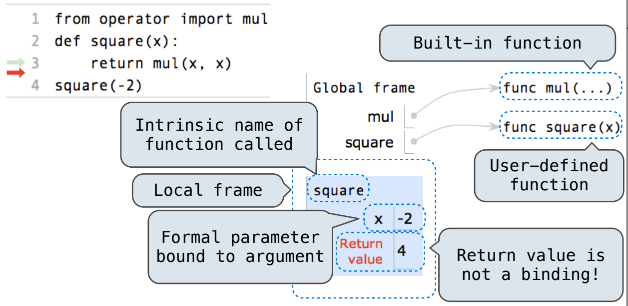
Statements and expressions

Red arrow points to next line.
Gray arrow points to the line just executed

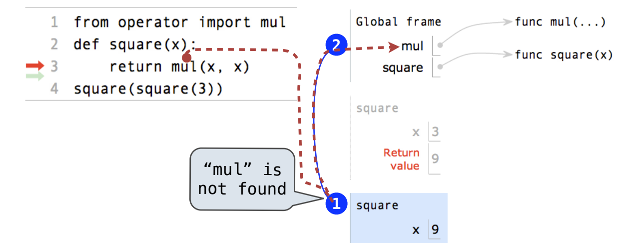
Frames (right):

A name is bound to a value

In a frame, there is at most one binding per name



A name evaluates to the value bound to that name in the earliest frame of the current environment in which that name is found.



Evaluation rule for call expressions:

1. Evaluate the operator and operand subexpressions.
2. Apply the function that is the value of the operator subexpression to the arguments that are the values of the operand subexpressions.

Applying user-defined functions:

1. Create a new local frame with the same parent as the function that was applied.
2. Bind the arguments to the function's formal parameter names in that frame.
3. Execute the body of the function in the environment beginning at that frame.

Execution rule for def statements:

1. Create a new function value with the specified name, formal parameters, and function body.
2. Its parent is the first frame of the current environment.
3. Bind the name of the function to the function value in the first frame of the current environment.

Execution rule for assignment statements:

1. Evaluate the expression(s) on the right of the equal sign.
2. Simultaneously bind the names on the left to those values, in the first frame of the current environment.

Execution rule for conditional statements:

Each clause is considered in order.
1. Evaluate the header's expression.
2. If it is a true value, execute the suite, then skip the remaining clauses in the statement.

Evaluation rule for or expressions:

1. Evaluate the subexpression <left>.
2. If the result is a true value v, then the expression evaluates to v.
3. Otherwise, the expression evaluates to the value of the subexpression <right>.

Evaluation rule for and expressions:

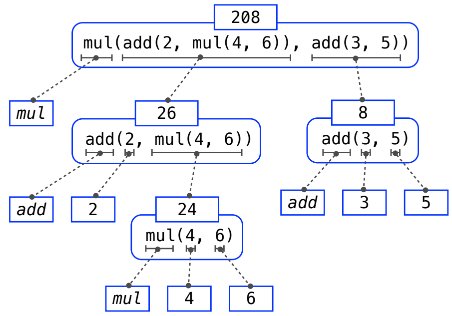
1. Evaluate the subexpression <left>.
2. If the result is a false value v, then the expression evaluates to v.
3. Otherwise, the expression evaluates to the value of the subexpression <right>.

Evaluation rule for not expressions:

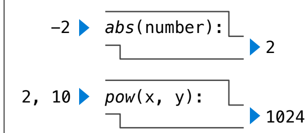
1. Evaluate <exp>; The value is True if the result is a false value, and False otherwise.

Execution rule for while statements:

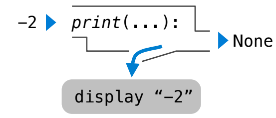
1. Evaluate the header's expression.
2. If it is a true value, execute the (*whole*) suite, then return to step 1.

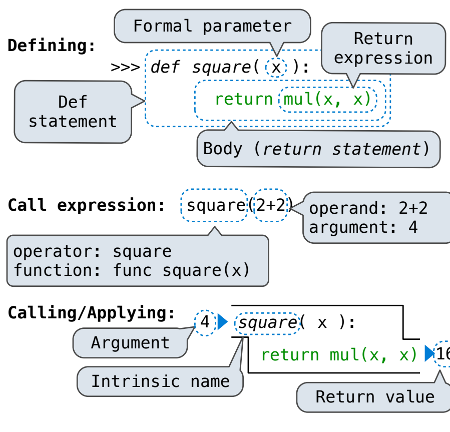


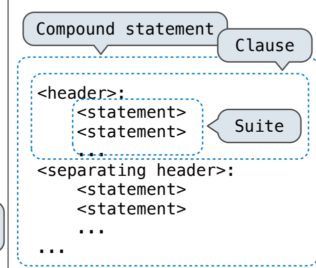
Pure Functions

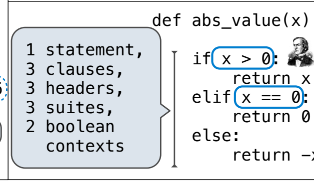


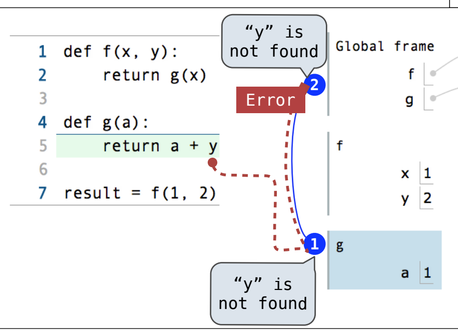
Non-Pure Functions



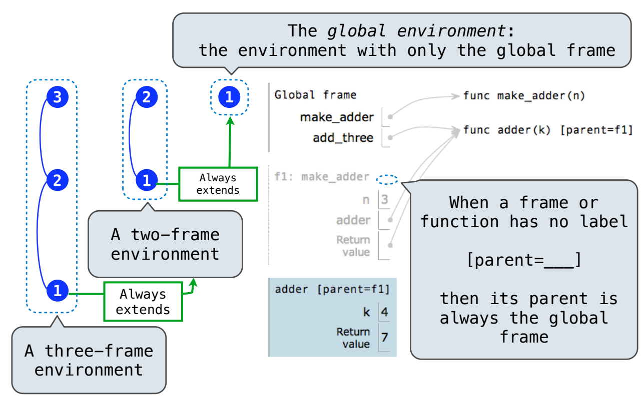








- An environment is a sequence of frames
- An environment for a non-nested function (no def within def) consists of one local frame, followed by the global frame



A frame *extends* the environment that begins with its parent

```
def cube(k):
    return pow(k, 3)

def summation(n, term):
    """Sum the first n terms of a sequence.

    >>> summation(5, cube)
    225
    """
    total, k = 0, 1
    while k <= n:
        total, k = total + term(k), k + 1
    return total
```

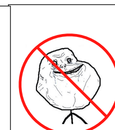
Function of a single argument (not called term)

A formal parameter that will be bound to a function

The cube function is passed as an argument value

$0 + 1^3 + 2^3 + 3^3 + 4^3 + 5^5$

The function bound to term gets called here



Higher-order function: A function that takes a function as an argument value or returns a function as a return value

Nested def statements: Functions defined within other function bodies are bound to names in the local frame

```
square = lambda x,y: x * y
```

- *Evaluates to a function. No "return" keyword!*
- A function
- with formal parameters x and y
- that returns the value of "x * x"
- Must be a single expression

```
def make_adder(n):
    """Return a function that takes one argument k and returns k + n.

    >>> add_three = make_adder(3)
    >>> add_three(4)
    7
    """
    def adder(k):
        return k + n
    return adder
```

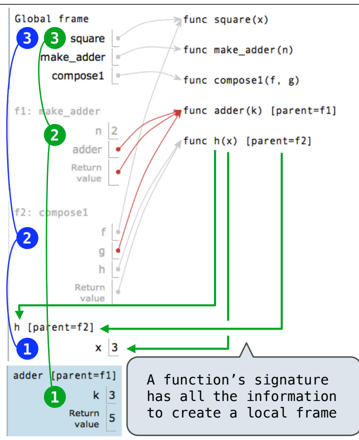
- A function that returns a function
- The name add_three is bound to a function
- A local def statement
- Can refer to names in the enclosing function

```
def square(x):
    return x * x

def make_adder(n):
    def adder(k):
        return k + n
    return adder

def compose1(f, g):
    def h(x):
        return f(g(x))
    return h

compose1(square, make_adder(2))(3)
```



- Every user-defined **function** has a *parent frame* (often global)
- The parent of a **function** is the frame in which it was *defined*
- Every local **frame** has a *parent frame* (often global)
- The parent of a **frame** is the parent of the function *called*

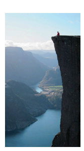
```
def curry2(f):
    """Returns a function g such that g(x)(y) returns f(x, y)."""
    def g(x):
        def h(y):
            return f(x, y)
        return h
    return g
```

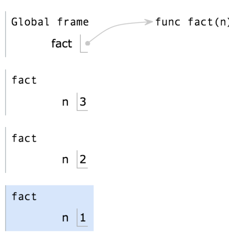
Currying: Transforming a multi-argument function into a single-argument, higher-order function.

- The **def statement header** is similar to other functions
- Conditional statements check for **base cases**
- Base cases are evaluated **without recursive calls**
- Recursive cases are evaluated **with recursive calls**

```
def sum_digits(n):
    """Return the sum of the digits of positive integer n."""
    if n < 10:
        return n
    else:
        all_but_last, last = n // 10, n % 10
        return sum_digits(all_but_last) + last
```

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)
```



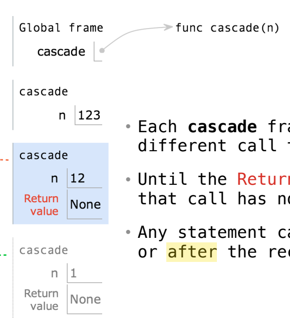


Is fact implemented correctly?

1. Verify the base case.
2. Treat fact as a functional abstraction!
3. Assume that fact(n-1) is correct.
4. Verify that fact(n) is correct, assuming that fact(n-1) correct.

```
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)

cascade(123)
```



Program output:

```
123
12
1
12
```

- Each **cascade** frame is from a different call to **cascade**.
- Until the Return value appears, that call has not completed.
- Any statement can appear before or after the recursive call.

```
square = lambda x: x * x      VS      def square(x):
                                          return x * x
```

- Both create a function with the same domain, range, and behavior.
- Both functions have as their parent the environment in which they were defined.
- Both bind that function to the name square.
- Only the def statement gives the function an intrinsic name.

When a function is defined:

1. Create a **function value**: func *<name>*(*<formal parameters>*)
2. If the **parent frame** of that function is not the global frame, add matching *labels* to the **parent frame** and the **function value** (such as *f1*, *f2*, or *f3*).

   f1: make_adder func adder(k) [parent=f1]

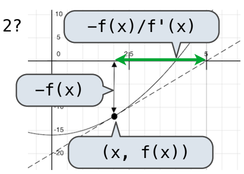
3. Bind *<name>* to the **function value** in the first frame of the current environment.

When a function is called:

1. Add a **local frame**, titled with the *<name>* of the function being called.
2. If the function has a parent label, copy it to the **local frame**: [parent=*<label>*]
3. Bind the *<formal parameters>* to the arguments in the **local frame**.
4. Execute the body of the function in the environment that starts with the **local frame**.

How to find the square root of 2?

```
>>> f = lambda x: x*x - 2
>>> df = lambda x: 2*x
>>> find_zero(f, df)
1.4142135623730951
```



Begin with a function f and an initial guess x

1. Compute the value of f at the guess: f(x)
2. Compute the derivative of f at the guess: f'(x)
3. Update guess to be: $x - \frac{f(x)}{f'(x)}$

```
def improve(update, close, guess=1):
    """Iteratively improve guess with update until close(guess) is true."""
    while not close(guess):
        guess = update(guess)
    return guess

def approx_eq(x, y, tolerance=1e-15):
    return abs(x - y) < tolerance

def find_zero(f, df):
    """Return a zero of the function f with derivative df."""
    def near_zero(x):
        return approx_eq(f(x), 0)
    return improve(newton_update(f, df), near_zero)

def newton_update(f, df):
    """Return an update function for f with derivative df,
    using Newton's method."""
    def update(x):
        return x - f(x) / df(x)
    return update

def power(x, n):
    """Return x * x * x * ... * x for x repeated n times."""
    product, k = 1, 0
    while k < n:
        product, k = product * x, k + 1
    return product

def nth_root_of_a(n, a):
    """Return the nth root of a."""
    def f(x):
        return power(x, n) - a
    def df(x):
        return n * power(x, n-1)
    return find_zero(f, df)
```

- Recursive decomposition: finding simpler instances of the problem: **partition(6, 4)**
- Explore two possibilities:
  - Use at least one 4
  - Don't use any 4
- Solve two simpler problems:
  - **partition(2, 4)**
  - **partition(6, 3)**
- Tree recursion often involves exploring different choices.

```
def count_partitions(n, m):
    if n == 0:
        return 1
    elif n < 0:
        return 0
    elif m == 0:
        return 0
    else:
        with_m = count_partitions(n-m, m)
        without_m = count_partitions(n, m-1)
        return with_m + without_m
```

```
from operator import floordiv, mod
def divide_exact(n, d):
    """Return the quotient and remainder of dividing N by D.

    >>> q, r = divide_exact(2012, 10)
    >>> q
    201
    """
    return floordiv(n, d), mod(n, d)
```

- Multiple assignment to two names
- Multiple return values, separated by commas