Here we are going to have a look at what spin really *is* — or at least give a hint at where it comes from mathematically, which obviously is not quite the same thing philosophically speaking, but as it turns out may be the best we can ever hope to do. This worksheet is probably too long and ambitious to get through in class, but read and think about the rest of it at home if you find it interesting, and ask me in person or by e-mail anytime if you are curious and want help to understand the full picture.

We start off by assuming that we have a quantum point particle which is described simply by specifying its position (forget about spin or other luxury ornamentation for the moment), so that the quantum state is fully described by a simple position space wave function $\psi(\mathbf{r})$.

1. Suppose we rotate the system by a tiny angle ϵ around the z-axis. Under this operation, the following happens to the x and y coordinates (to first order in ϵ):

$$x \to x - \epsilon y \qquad\qquad y \to y + \epsilon x \tag{1}$$

   Given this, how does the wave function change to first order in ϵ? (Hint: Taylor expand to first order)

2. From the previous question, you should have found that the first-order change in the wave function under the rotation is

$$\psi(\mathbf{r}) \to \psi(\mathbf{r}) + \epsilon \left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right] \psi(\mathbf{r}) \tag{2}$$

   For this reason, one often says that rotations around the z-axis are *generated* by the operator $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Now if you use the fact that the momentum operator in the x-direction in position representation is given by $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$, and similarly in the y- and z-direction, what does this generating operator look like if you put in momentum operators where appropriate? What physical observable operator does this remind you of (Hint: think about angular momenta…)

3. Now suppose we rotate through a finite and not-so-tiny angle θ. What is the operator $\hat{R}_z(\theta)$ which creates the corresponding change in the wave function, i.e. if $\psi(\mathbf{r}) \to \hat{R}_z(\theta)\psi(\mathbf{r})$ under the rotation, what is $\hat{R}_z(\theta)$? Hint: either remember something from the first homework, or do the rotation in many tiny steps by rotating N times through a small angle $\epsilon = \theta/N$, each time acting on the state with $1 - \frac{i}{\hbar}\frac{\theta}{N}\hat{L}_z$, and in the end let $N \to \infty$. Your answer should hopefully involve an exponential. Looks familiar?

4. What kind of operator is $\hat{R}_z$? (Hint: do distinguishable states become less distinguishable or vice versa just because we rotate our system?) What kind of operator is the operator $\hat{L}_z$ that "generates" it? Your conclusion should apply to any transformation operator and its generator that are involved in doing a transformation in which we don't measure anything or let any information "leak" out of the system.

If we now generalize from the z-axis to any arbitrary axis $\hat{\mathbf{n}}$, the take-home message from the previous questions should be that the effect of rotating the system an angle θ around the axis $\hat{\mathbf{n}}$ is to act on the wave function with the operator

$$\hat{R}_{\hat{\mathbf{n}}}(\theta) = e^{-\frac{i}{\hbar}\theta\hat{L}_{\hat{\mathbf{n}}}} \tag{3}$$

where $\hat{L}_{\hat{\mathbf{n}}}$ is the operator for the angular momentum component along the $\hat{\mathbf{n}}$-axis, which is Hermitian so that $R_{\hat{\mathbf{n}}}(\theta)$ is unitary. By considering how the system would interact with a measurement device in experiments where you measure the $\hat{\mathbf{n}}$-component of the physical angular momentum of the particle, it is also possible to aruge at least handwavingly that the only values you can obtain in a precise measurement are in fact given by the eigenvalues of $\hat{L}_{\hat{\mathbf{n}}}$, so the operator $\hat{L}_{\hat{\mathbf{n}}}$ really is *the* operator for angular momentum along the $\hat{\mathbf{n}}$-direction. The angular momentum operator along some axis is in other words the operator which generates rotations around that axis

Now let us consider particles that have more degrees of freedom than just their spatial position, and which can therefore not be described by just a scalar spatial wave function $\psi(\mathbf{r})$. We can then usually represent their state by adding one or more *indices* to the wave function, which represent the possible values of the extra degrees of freedom, but for now we will not make any assumption on what we specifically need to do to the wave function and just represent the state of the particle abstractly in Dirac notation with the ket $|\psi\rangle$.

When we now rotate the system an angle θ around some axis $\hat{\mathbf{n}}$, we will need two operators to change the state, first $\hat{R}_{\hat{\mathbf{n}}}(\theta) = \exp(-\frac{i}{\hbar}\theta\hat{L}_{\hat{\mathbf{n}}})$ to take care of the spatially dependent part of $|\psi\rangle$, and then a new operator $\hat{U}_{\hat{\mathbf{n}}}(\theta)$ to transform the part of the state that depends on the new non-spatial degree of freedom.

Just like $\hat{R}_{\hat{\mathbf{n}}}(\theta)$, $\hat{U}_{\hat{\mathbf{n}}}(\theta)$ has to be a unitary operator, and since θ can vary continuously, we assume that $\hat{U}_{\hat{\mathbf{n}}}$ is also generated by some Hermitian operator which we will call $\hat{S}_{\hat{\mathbf{n}}}$, i.e.

$$\hat{U}_{\hat{\mathbf{n}}}(\theta) = e^{-\frac{i}{\hbar}\theta\hat{S}_{\hat{\mathbf{n}}}} \tag{4}$$

The new state after the rotation is now $\hat{U}_{\hat{\mathbf{n}}}(\theta)\hat{R}_{\hat{\mathbf{n}}}(\theta)|\psi\rangle$.

5. What is now the Hermitian operator which generates the *whole* transformation of $|\psi\rangle$, both the space-dependent part and the new degree of freedom? (Hint: remember that since $\hat{L}_{\hat{\mathbf{n}}}$ and $\hat{S}_{\hat{\mathbf{n}}}$ act on different things, they commute, and then think about what $e^{\hat{A}}e^{\hat{B}}$ is equal to when $\hat{A}$ and $\hat{B}$ commute) It is possible to show that this new observable is a conserved quantity, but the "old" angular momentum is not necessarily, so this change in what Hermitian operator generates the full rotation of the state is important for keeping track of what happens in physical processes.

6. A conclusion that you can draw from the previous question should be that the total "angular momentum" of the particle along the $\hat{\mathbf{n}}$-direction is now no longer just the orbital angular momentum corresponding to $\hat{L}_{\hat{\mathbf{n}}}$, but that you need to add to this some new observable corresponding to the operator $\hat{S}_{\hat{\mathbf{n}}}$. As you have perhaps guessed, this could e.g. be the infamous "intrinsic spin" of the electron that we have been throwing around in class all the time. So what does this "spin" physically correspond to? (think long and hard about this, and come talk to me if you want to be sure about the answer, though I should warn you that the honest answer may seem a little disappointing at first...) Is there some way to decide whether $\hat{S}_{\hat{\mathbf{n}}}$ has to do with something physically spinning or circulating in real space? Is there any way we can find out for sure where this new observable actually comes from? What might limit us?