C/CS/Phys C191 **2 Qubit gates, EPR, Bell's inequality** 9/8/05
Fall 2005 Lecture 4

1 Two-qubit gate: CNOT

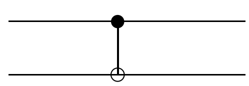
The controlled-not (CNOT) gate exors the first qubit into the second qubit ($|a,b\rangle \rightarrow |a, a\oplus b\rangle = |a, a+b \bmod 2\rangle$). Thus it permutes the four basis states as follows:

$$\begin{array}{ll} 00 \rightarrow 00 & 01 \rightarrow 01 \\ 10 \rightarrow 11 & 11 \rightarrow 10 \ . \end{array}$$

As a unitary 4×4 matrix, the CNOT gate is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

In a quantum circuit diagram, the CNOT gate has the following representation. The upper wire is called the control bit, and the lower wire the target bit.



It turns out that this is the only two qubit gate we need to think about ...

2 Bell states (EPR pairs)

There are four Bell states:

$$\begin{aligned} |\Phi^{\pm}\rangle &= \tfrac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) \\ |\Psi^{\pm}\rangle &= \tfrac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right) \ . \end{aligned}$$

These are maximally entangled states on two qubits. They cannot be product states because there are no cross terms.

Consider one of the Bell states (also known as a EPR pair):

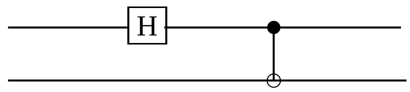
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Measuring the first qubit of $|\Psi^-\rangle$ in the standard basis yields a 0 with probability 1/2, and 1 with probability 1/2. Likewise, measuring the second qubit of $|\Psi^-\rangle$ yields the same outcomes with the same probabilities. Thus measuring one, and only one, qubit of this state yields a perfectly random outcome.

However, determining either qubit *exactly* determines the other. For example, if qubit 1 is measured and gives a 0, this projects the Bell state onto the state $\left|01\right\rangle$ and the second qubit is then definitely a 1.

Furthermore, measurement of $\left|\Psi^-\right\rangle$ in any basis will yield opposite outcomes for the two qubits. To see this, check that $\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(\left|vv^\perp\right\rangle - \left|v^\perp v\right\rangle\right)$, for any $\left|v\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$, $\left|v^\perp\right\rangle = \bar{\alpha}\left|1\right\rangle - \bar{\beta}\left|0\right\rangle$.

We can generate the Bell states with a Hadamard gate and a CNOT gate. Consider the following diagram:



The first qubit is passed through a Hadamard gate and then both qubits are entangled by a CNOT gate.

If the input to the system is $|0\rangle \otimes |0\rangle$, then the Hadamard gate changes the state to

$$\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|10\rangle \quad ,$$

and after the CNOT gate the state becomes $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the Bell state $|\Phi^+\rangle$. In fact, one can verify that the four possible inputs produce the four Bell states:

$$|00\rangle \mapsto \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle; \qquad |01\rangle \mapsto \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Psi^+\rangle;$$
$$|10\rangle \mapsto \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\Phi^-\rangle; \qquad |11\rangle \mapsto \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\Psi^-\rangle.$$

3 EPR Paradox

In 1935, Einstein, Podolsky and Rosen (EPR) wrote a paper ”Can quantum mechanics be complete?” [Phys. Rev. 47, 777, Available online via PROLA: `http://prola.aps.org/abstract/PR/v47/i10/p777_1`]

For example, consider coin-flipping. We can model coin-flipping as a random process giving heads 50% of the time, and tails 50% of the time. This model is perfectly predictive, but incomplete. With a more accurate experimental setup it would in principle be possible to follow the dynamics of the coin and to determine precisely the range of initial parameters for which the coin ends up heads, and the range for which it ends up tails.

We saw above that for a Bell state, when you measure first qubit, the second qubit is completely determined. However, if two qubits are far apart, then the second qubit must have had a determined state in some time interval *before* measurement, since the speed of light is finite. Moreover this holds in any basis. This appears analogous to the coin flipping example, i.e., there might be a more complete description which allows the qubit states to be predicted. EPR therefore suggested that there is a more complete theory where“God does not throw dice”.

EPR made two assumptions:

i) reality principle - the values of physical quantities have physical reality independent of whether a measurement of them is made or not.

ii) locality principle - the result of a measurement on one system cannot influence the result of a measurement on the second system

These two assumptions give rise to a contradiction, nicely illustrated by the analysis of the two qubits in a Bell state, due to Bohm. See Benenti et al., Sec. 2.5. A source emits the Bell state

$$\left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}(\left|01\right\rangle - \left|10\right\rangle)$$

and sends one qubit to Alice, and one to Bob. If Alice measures her qubit in the standard basis and e.g., gets a 1, then Bob will get a 0 upon measuring in the standard basis. On the other hand, if Alice measures her qubit in the Hadamard basis $\{\left|+\right\rangle, \left|-\right\rangle\}$, and gets a +, then Bob will get a - in the Hadamard basis. However, the states $\left|0\right\rangle$ and $\left|-\right\rangle$ are not the same, they differ by a Hadamard rotation. Which state Bob ends up with depends on the measurement made by Alice. This contradicts the assumption of locality. EPR concluded that such situations imply that quantum mechanics is not a complete theory of the physical world.

What would a more complete theory look like? Here is the most extravagant framework... When the entangled state is created, the two particles each make up a (very long!) list of all possible experiments that they might be subjected to, and decide how they will behave under each such experiment. When the two particles separate and can no longer communicate, they consult their respective lists to coordinate their actions. To describe such behavior one would have to invoke the existence of 'local hidden variables' that are not evident in the quantum description.

It was not until 1964, almost three decades later, that a verifiable and quantitative measure of the local realism assumption was provided. This was given by Bell, who constructed correlation functions of the measurements of Alice and Bob that satisfy a strict inequality under the assumptions of local realism. However, the quantum analog of the correlation functions can violate the inequality for certain choices of measurement basis. The Bell inequality was subsequently tested experimentally in 1981 by Aspect and co-workers, using Bell (EPR) pairs constructed from photon polarization states. Aspect et al. found that indeed nature does not obey the Bell inequalities and so violates local realism. This is consistent with the predictions of quantum mechanics for EPR pairs summarized above and supports the view that nothing *can* be known about the quantum state until a measurement is made. It also tells us that the quantum correlations in an EPR pair are 'stronger' than classical correlations. A detailed analysis of the Bell inequality for $\left|\Psi^-\right\rangle$ can be found at http://minty.caltech.edu/Ph195/downloads.htm (lecture 10/24, pp. 11-15).

For further reading on the EPR paradox, Bell's inequality, and the experimental verification of violation of this by quantum systems, see

1. Styer, Ch. 6

2. D. Mermin, Physics Today vol. 38(4), April 1985, pp. 38-47.

Nonlocality and Bell Inequalities

(Based on the discussion in Chris Isham's book, *Lectures on Quantum Theory: Mathematical and Structural Foundations* (Imperial College Press, 1995).)

Say we have two experimenters, Alice and Bob, whose labs are located many kilometers apart. Their labs are basically identical, actually, each consisting of one particle 'detector' that has one meter, one switch, and a bell. The meter is for reading out the result of a measurement (which we assume to be either ± 1), while the switch is used to select which of two types of measurements the experimenter would like to make. On Alice's side we'll label the two possibilities A and A', and on Bob's side B and B'. The bell rings each time a particle hits the detector, letting the experimenter know when he or she can read out the result of his/her selected measurement.

So where do these particles come from? Midway between Alice's lab and Bob's there is a 'pair source.' This source always produces particles in pairs, sending one to Alice and the other to Bob. We assume that the particles have some internal degree of freedom, which is what Alice's and Bob's detectors are designed to measure. The pair source prepares the internal states of the particles in some unknown, possibly random fashion.

The 'experiment' consists of the following procedure. The source prepares and emits one pair of particles per unit of time, so Alice and Bob know that they may expect to receive particles at a regular rate. Once per unit time, they each (independently) select a random setting for their switch, wait for their bell to ring, and then read off and write down the measurement result.

Hence after ten rounds, *e.g.*, Alice's and Bob's lab books might look something like this:

Alice		**Bob**	
A	-1	B'	-1
A	$+1$	B'	-1
A'	$+1$	B	$+1$
A	$+1$	B	$+1$
A'	-1	B	-1
A'	-1	B'	$+1$
A	$+1$	B	-1
A'	-1	B'	$+1$
A	$+1$	B'	$+1$
A	-1	B	$+1$

1

Although this experimental scenario seems extremely general, it turns out that we have already specified enough to derive some important predictions about the statistics of Alice's and Bob's measurement records!

Let's start by making some reasonable assumptions about the overall behavior of the experiment:

1. ***Local determinism*** – we might like to believe that the result of Alice's measurement (either A or A') is *locally* determined by the physical state of the particle she receives from the pair source. It should not depend on the state of Bob's particle, since in this scenario Bob could be really far away! And the result of Alice's measurement certainly should not depend on Bob's choice of measurement – that is, whether Alice's meter reads $+$ or -1 should not depend on whether Bob has his switch set to B or B'...
2. ***Objective reality*** – Even though Alice (and Bob) must choose to make one measurement or the other (A or A') on any given particle, each particle 'knows' what its value is for both measurements. That is, sufficient information to determine the outcome of either measurement is encoded in the internal state of each particle.

Under these assumptions, we can write down the following model for this experiment. In each round, the pair souce produces a pair of particles with the following information encoded in their internal states:

$$A_n = \pm 1, \quad A'_n = \pm 1, \quad B_n = \pm 1, \quad B'_n = \pm 1. \qquad 2$$

Here the four possible measurement labels are treated as random variables, with the subscript labelling the round. As a logical consequence of local determinism and objective realism, we can assume the existence of a *joint probability distribution* $P(A, A', B, B')$. Hence, it should be meaningful to consider correlation functions of all four random variables simultaneously, and these correlation functions should be measurable by Alice and Bob.

Consider the following function of the random variables,

$$g_n = A_n B_n + A'_n B_n + A_n B'_n - A'_n B'_n. \qquad 3$$

Were we to tabulate the 16 possible values of g_n, we would magically find that $g_n = \pm 2$. However, an easier way to see this is to note that the last term in the sum is equal to the product of the first three, since $A_n^2 = (A'_n)^2 = B_n^2 = (B'_n)^2 = +1$:

$$\begin{aligned} A'_n B'_n &= (A_n B_n)(A'_n B_n)(A_n B'_n) \\ &= A_n^2 B_n^2 A'_n B'_n. \end{aligned} \qquad 4$$

Then if $A'_n B'_n = +1$, the set $\{A_n B_n, A'_n B_n, A_n B'_n\}$ has either zero or two -1's, hence $g_n = A_n B_n + A'_n B_n + A_n B'_n - A'_n B'_n$ must be either $+2$ or -2. If on the other hand $A'_n B'_n = -1$, the set must have either zero or two $+1$'s, hence g_n must be either -2 or $+2$.

In any case, it follows that

$$\left|\frac{1}{N}\sum_{n=1}^{N} g_n\right| = \frac{1}{N}\left|\sum_{n=1}^{N} A_n B_n + \sum_{n=1}^{N} A_n' B_n + \sum_{n=1}^{N} A_n B_n' - \sum_{n=1}^{N} A_n' B_n'\right| \leq 2. \qquad 5$$

This is one form (due to Clauser, Horne, Shimony, and Holt) of Bell's famous inequality.

It should be noted that at this point, all we have relied on in our derivation is basic probability theory! Hence the Bell Inequality is a ***model-independent*** prediction about measurement statistics in a world that is locally deterministic and allows objective realism.

Hence experimental violations of the Inequality actually tell us something about Nature, not just quantum theory!

As it turns out, one can actually go to the lab and perform experiments of precisely the type described above, and find that this inequality is strongly violated! For example, see

- G. Weihs *et al*, "Violation of Bell's Inequality under Strict Einstein Locality Conditions," Phys. Rev. Lett. **81**, 5039-5043 (1998);
- W. Tittel *et al*, "Violation of Bell Inequalities by Photons More Than 10 km Apart," Phys. Rev. Lett. **81**, 3563-3566 (1998);
- A. Aspect, "Bell's inequality test: more ideal than ever," Nature **398**, 189-190 (1999).

In experiments of this type, the key is to construct a source that produces pairs of photons an *entangled* state such as

$$|\Psi_{ab}\rangle = \frac{1}{\sqrt{2}}(|0_a 1_b\rangle - |1_a 0_b\rangle). \qquad 6$$

In each round of the experiment, Alice's two measurements correspond to the observables $\mathbf{A} = \boldsymbol{\sigma}_z^a$ and $\mathbf{A}' = \cos\phi\,\boldsymbol{\sigma}_z^a + \sin\phi\,\boldsymbol{\sigma}_x^a$, where

$$\begin{aligned}\boldsymbol{\sigma}_z^a &= |0_a\rangle\langle 0_a| - |1_a\rangle\langle 1_a|,\\ \boldsymbol{\sigma}_x^a &= |0_a\rangle\langle 1_a| + |1_a\rangle\langle 0_a|.\end{aligned} \qquad 7$$

On Bob's side we choose $\mathbf{B} = \boldsymbol{\sigma}_z^b$ and $\mathbf{B}' = \cos\phi\,\boldsymbol{\sigma}_z^b - \sin\phi\,\boldsymbol{\sigma}_x^b$. The eigenvalues of **A** and **B** are clearly ± 1, and it turns out that those of $\mathbf{A}'$ and $\mathbf{B}'$ are also ± 1. For example, the eigenstates of $\cos\phi\,\boldsymbol{\sigma}_z + \sin\phi\,\boldsymbol{\sigma}_x$ are simply

$$\begin{aligned}|\tilde{0}\rangle &= \cos\frac{\phi}{2}|0\rangle + \sin\frac{\phi}{2}|1\rangle,\\ |\tilde{1}\rangle &= \sin\frac{\phi}{2}|0\rangle - \cos\frac{\phi}{2}|1\rangle.\end{aligned} \qquad 8$$

Hence $\mathbf{A}'$ corresponds to projectors on a basis that is rotated from that of **A** by an angle $\phi/2$ (and similarly a rotation of $-\phi/2$ for $\mathbf{B}, \mathbf{B}'$).

Now we can compute the necessary correlation functions using the standard quantum probability rules:

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} A_n B_n &= \langle \mathbf{A}\otimes\mathbf{B}\rangle \\ &= \left\langle \mathbf{P}_0^a\mathbf{P}_0^b\right\rangle + \left\langle \mathbf{P}_1^a\mathbf{P}_1^b\right\rangle - \left\langle \mathbf{P}_0^a\mathbf{P}_1^b\right\rangle - \left\langle \mathbf{P}_1^a\mathbf{P}_0^b\right\rangle \\ &= -1.\end{aligned} \quad 9$$

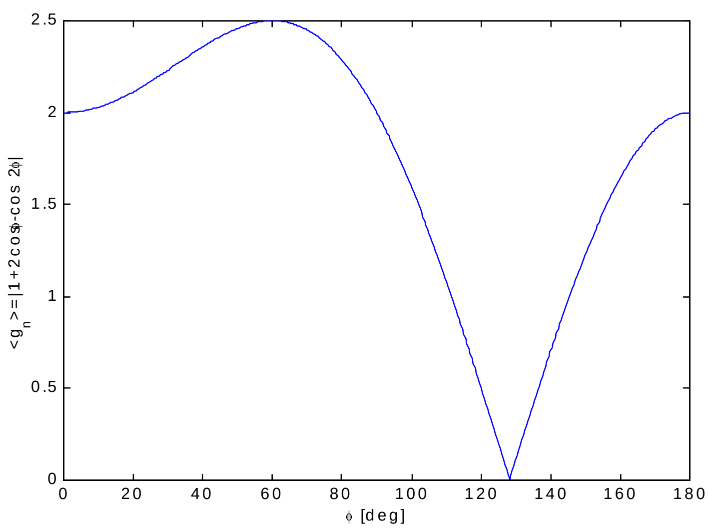
Similarly,

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} A_n B_n' &= \langle \mathbf{P}_0^a \cos\phi\, \boldsymbol{\sigma}_z^b\rangle - \langle \mathbf{P}_0^a \sin\phi\, \boldsymbol{\sigma}_x^b\rangle - \langle \mathbf{P}_1^a \cos\phi\, \boldsymbol{\sigma}_z^b\rangle + \langle \mathbf{P}_1^a \sin\phi\, \boldsymbol{\sigma}_x^b\rangle \\ &= -\frac{1}{2}\cos\phi - \frac{1}{2}\cos\phi = -\cos\phi. \\ \frac{1}{N}\sum_{n=1}^{N} A_n' B_n &= \left\langle \mathbf{P}_0^b \cos\phi\, \boldsymbol{\sigma}_z^a\right\rangle + \left\langle \mathbf{P}_0^b \sin\phi\, \boldsymbol{\sigma}_x^a\right\rangle - \left\langle \mathbf{P}_1^b \cos\phi\, \boldsymbol{\sigma}_z^a\right\rangle - \left\langle \mathbf{P}_1^b \sin\phi\, \boldsymbol{\sigma}_x^a\right\rangle \\ &= -\cos\phi. \\ \frac{1}{N}\sum_{n=1}^{N} A_n' B_n' &= \langle \cos^2\phi\, \boldsymbol{\sigma}_z^a\boldsymbol{\sigma}_z^b\rangle + \langle \cos\phi\sin\phi\, \boldsymbol{\sigma}_z^a\boldsymbol{\sigma}_x^b\rangle - \langle \cos\phi\sin\phi\, \boldsymbol{\sigma}_x^a\boldsymbol{\sigma}_z^b\rangle \\ &\quad - \left\langle \sin^2\phi\, \boldsymbol{\sigma}_x^a\boldsymbol{\sigma}_x^b\right\rangle \\ &= \frac{\cos^2\phi}{2}(-1-1) - \frac{\sin^2\phi}{2}(-1-1) = \sin^2\phi - \cos^2\phi \\ &= -\cos 2\phi.\end{aligned} \quad 10$$

Finally, we can construct the overall quantity

$$\begin{aligned}\frac{1}{N}\left|\sum_{n=1}^{N} g_n\right| &= |-1 - 2\cos\phi + \cos 2\phi| \\ &= |1 + 2\cos\phi - \cos 2\phi|.\end{aligned} \quad 11$$

Plotting this, we find that the Bell Inequality is violated ($\langle g_n\rangle > 2$) for $0 < \phi < 90°$:



So what's going on here? From the graph we see that our Bell Inequality can be violated when the two possible measurements that Alice and Bob can perform correspond to projections on nonorthogonal bases. Hence what is being exploited here is the extra-strong "quantum correlation" between two particles that have been prepared in an entangled state such as

$$|\Psi_{ab}\rangle = \frac{1}{\sqrt{2}}(|0_a 1_b\rangle - |1_a 0_b\rangle). \qquad 12$$