C/CS/Phys C191 **Qubits, Measurements, Notation** 9/01/05
Fall 2005 Lecture 2

1 Qubits

The basic entity of quantum information is a qubit (pronounced "cue-bit"), or a quantum bit. This corresponds to a 2-state quantum system. The quantum state of the system can be written as the linear superposition (column) vector $\binom{\alpha}{\beta} \in \mathscr{C}^2$. The meaning of this linear superposition is that the qubit is in the state (0) with probability amplitude $\alpha \in \mathscr{C}$ and in the excited state (1) with probability amplitude $\beta \in \mathscr{C}$. We can refer to 0 and 1 as the basis for the quantum state. It is as though the qubit "does not make up its mind" as to which of the 2 basis states it is in.

In Dirac notation, the qubit state may be written as:

$$\left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle \quad \alpha,\beta \in \mathscr{C} \quad \text{and} \quad |\alpha|^2 + |\beta|^2 = 1$$

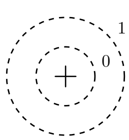
The Dirac notation has the advantage that it labels the basis vectors explicitly. This is very convenient because the notation expresses both that the state of the qubit is a vector, and that it is data (0 or 1) to be processed. The $\{\left|0\right\rangle, \left|1\right\rangle\}$ basis is called the standard or computational basis.

In general a column vector—called a "ket"— is denoted by $\left|\ \right\rangle$ and a row vector is —called a "bra"— is denoted by $\left\langle\ \right|$.

We now give three examples of physical realizations of qubits, but there are many more.

Energy levels of hydrogen atom

Consider the electron in a hydrogen atom. It can be in its ground state (i.e. an *s* orbital) or in an excited state. If this were a classical system, we could store a bit of information in the state of the electron: ground = 0, excited = 1. So we can also store a qubit of information in the quantum state of the electron, i.e., in the superposition $\left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$. Note that the electron actually has an infinite number of energy levels (indexed by quantum number n, with $E_n \propto -1/n^2$), but that as long as we can isolate two of them, we can use these two as a qubit.



Photon Polarization

There is a qubit associated with photon - its polarization. Recall that a photon moving along the z-axis has an associated electric field in the x-y plane. The frequency of the field is determined by the frequency of the photon. However, this still leaves the x-y components of the electric field unspecified. The 2-dimensional quantity specifying this field is the polarization of the photon.

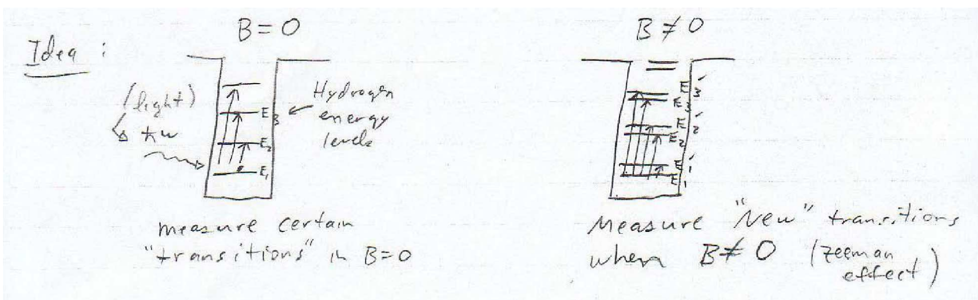
See notes on polarization on the web page Science Trek at http://www.colorado.edu/physics/2000.

Spin

Qubit systems can always be mapped onto an effective spin 1/2 system so it is important to understand what this is and where it comes from.

Elementary particles and composite particles carry an intrinsic angular momentum called spin. For our purposes, the most important particles are electrons and protons. To each of these is associated an angular momentum vector that can point up $|\uparrow>$ or down $|\downarrow>$. The quantum mechanical spin state of an electron or proton is thus $|\psi>=\alpha|\uparrow>+\beta|\downarrow>$. Therefore, spins can be used as qubits with $|0>=|\uparrow>$, $|1>=|\downarrow>$.

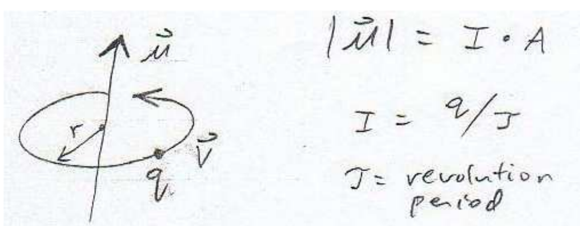
The spin angular momentum is intrinsic and signals the presence of an intrinsic magnetic moment. Uhlenbeck and Goudsmit introduced the concept of 'spin' in 1925 to explain the behavior of hydrogen atoms in a magnetic field:



The extra transitions can be explained if an electron has an *intrinsic* magnetic moment $\vec{\mu}$, since a magnetic moment in a magnetic field $\vec{B}$ has an energy $E=-\vec{\mu}\cdot\vec{B}$. In the context of QM, new energy levels can derive from $\vec{\mu}$ being oriented parallel or anti-parallel to $\vec{B}$.

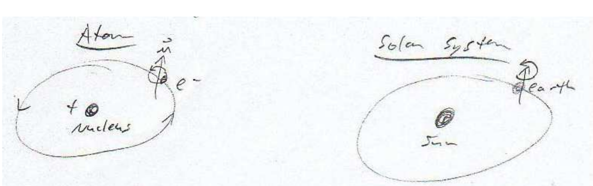
Where does $\vec{\mu}$ come from?

The simplest explanation is "classical": classically, a magnetic moment $\vec{\mu}$ comes from a loop of current.



The energy $E=-\vec{\mu}\cdot\vec{B}$ comes from $\vec{I}\times\vec{B}$ force of current in a B-field (Lorentz force). The lowest energy, and thereby the place where "the system wants to go", is obtained when the magnetic moment and B-field line up.

If an isolated electron has "intrinsic" $\vec{\mu}$ then the simplest explanation for this is that electron spins about some axis. This is independent of its orbital motion in an atom, just like the Earth's "spin" about the north pole is independent of its orbit around the sun.



Since $\vec{\mu}$ is associated with a "spinning" charge, then we can write $\vec{\mu}$ in terms of angular momentum. Anything that spins has angular momentum!

The simplest way to see this is classically for a spinning charge. For an electron the charge q is equal to $-e$. Angular momentum is given by $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$. $L = mvr$ for a charge of mass $= m$ moving in a a circle with velocity $= v$. The magnetic moment can be obtained as follows:

$$\mu = (current)\,(Area) = \frac{q}{\tau} \cdot \pi r^2$$

But the revolution period $\tau = \frac{2\pi r}{v}$. Substituting for τ and v in terms of L, we obtain

$$\vec{\mu} = \frac{q}{2m}\vec{L}$$

Now comes the tricky part. *The electron is not actually spinning about some axis!* It only acts as though it is. Electrons are point particles which, as far as we know, have no "size" in the traditional sense. Therefore the r in the previous discussion of spinning charge is not meaningful. The *intrinsic* angular momentum of an electron has nothing to do with "orbital" motion, but it does lead to an intrinsic $\vec{\mu}$. This is a relativistic effect that can be derived from the Dirac Equation (Relativistic Schrodinger equation for spin-$\frac{1}{2}$ particles), but it holds for electrons that are not moving fast.

This intrinsic angular momentum is called "spin" = $\vec{S}$.

For an electron, classically: $\vec{\mu} = -\frac{e}{2m}\vec{L}$, while quantum mechanically: $\vec{\mu} = -\frac{ge}{2m}\vec{S}$.

What is g? g is called the g-factor and it is a unitless correction factor due to QM. For electrons, $g \approx 2$. For protons, $g \approx 5.6$. You should also note that $\frac{m_{proton}}{m_{electron}} \approx 2000$, so we conclude that $\mu_{\vec{proton}} \ll \mu_{\vec{electron}}$.

So, to understand behavior of the electron's intrinsic magnetic moment $\vec{\mu}$ (which is an observable we can measure) then we must understand the behavior of its intrinsic angular momentum = $\vec{S}$. This is why spin is important. Since the electron is small, $\vec{S}$ must be described by QM.

See also notes "electrons in atoms" (look for "spin" section) on the web page Science Trek at http://www.colorado.edu/phy

1.1 The Bloch Sphere

A very nice way to think of the quantum states of qubits is via the "Bloch Sphere." This is a convenient mapping for all possible single-qubit states. See Figure 1 below.

θ and ϕ are the usual spherical coordinates, with $0 \leq \theta \leq \pi, 0 < \phi \leq 2\pi$. Every point on the sphere represents a possible qubit. All possible qubits (within an overall multiplicative phase factor) can be thought of as vectors on this unit sphere. A vector on the Bloch Sphere represents this qubit:

$$|\psi> = cos\frac{\theta}{2}|0> + e^{i\phi} sin\frac{\theta}{2}|1>$$

2 Measurement Revisited

This linear superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is part of the private world of the qubit. For us to know the state of the qubit, whether this is realized by an electron, a photon, or an electron spin, we must make a

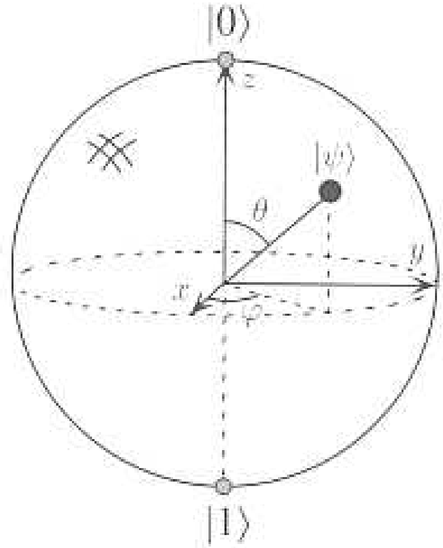


Figure 1: The Bloch sphere. $|0\rangle$ is at the North pole, $|1\rangle$ at the South pole.

measurement. Measuring $|\psi\rangle$ in the standard basis $\{|0\rangle, |1\rangle\}$ yields $|0\rangle$ with probability $|\alpha|^2$, and $|1\rangle$ with probability $|\beta|^2$.

One important aspect of the measurement process is that it alters the state of the quantum system: the effect of the measurement is that the new state is exactly the outcome of the measurement. I.e., if the outcome of the measurement is $|0\rangle$, then following the measurement, the qubit is in state $|0\rangle$. This implies that you cannot collect any additional information about the amplitudes α_j by repeating the measurement on the resuling state. You need to make many identical measurements on a set (ensemble) of equivalent states.

Repeated measurements on a state may however be useful for other reasons. We shall examine this with analysis of the measurement process for photon polarization. The polarization of a photon can be measured by using a polaroid or a calcite crystal. These materials act as filters that select only one component of the electric field vector. See the section on polarization in ScienceTrek at http://www.colorado.edu/physics/2000. A polaroid sheet (suitably oriented) transmits x-polarized photons $|x\rangle$ and absorbs y-polarized photons $|y\rangle$. Thus a photon that is in a superposition $|\phi\rangle = \alpha|x\rangle + \beta|y\rangle$ is transmitted with probability $|\alpha|^2$ if the polaroid sheet is oriented to transmit x and with probability $|\beta|^2$ is the sheet is oriented to transmit y. In the former case the final state is $|x\rangle$, in the latter case it is $|y\rangle$.

Consider passing a photon in state $|\psi\rangle$ through 2 polaroid filters, first an x filter, then a y filter. After the first filter we have $|x\rangle$ with prob. $|\alpha|^2$. After the second filter we have nothing, with prob. 1. Where has the photon gone? During passage through the first filter it was absorbed by the first filter with prob. $|\beta|^2$. If it got through this first filter, it was absorbed by the second filter with prob. 1. Note that the experiment may also be interpreted as the results of identical experiments on many identical photons in state $|\psi\rangle$.

Now consider what happens if we interpose a third polaroid sheet at a 45 degree angle between the first two. Now a photon that is transmitted by the first sheet makes it through the next two with probability $1/4$. Why is this? The polarization of light after the first filter is $|x\rangle$. The second filter is oriented at 45 degrees, i.e., it will pass photons with polarization orientation $\vec{v} = \frac{1}{\sqrt{2}}(\vec{x} + \vec{y})$. So let's express $|x\rangle$ in the basis $|v\rangle$ and its orthogonal complement $\left|v^{\perp}\right\rangle$. This is also known as the $\left|+\right\rangle, \left|-\right\rangle$ basis.

$$\left|x\right\rangle = \frac{1}{\sqrt{2}}\left(\left|v\right\rangle + \left|v^{\perp}\right\rangle\right)$$

Now the light passing through the first filter is in state $\left|x\right\rangle$ with probability $|\alpha|^2$. The probability this light passes the second filter is equal to the probability that a $\left|0\right\rangle$ qubit ends up in $\left|+\right\rangle$ when measured in the $\left|+\right\rangle, \left|-\right\rangle$ basis. Reading off from the above equation, we see that this probability is 1/2. Those photons that do pass successfully through the second filter now have a resulting polarization $\frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle)$. The probability of this state now passing the third filter oriented in y is then 1/2. What is the overall probability of having a photon pass successfully through all 3 sheets? Answer: $|\alpha|^2/4$, obtained by multiplying the three probabilities for successive passage. The final photon state is $\left|y\right\rangle$.

Note that one effect of these measurements is to effectively rotate the plane of polarization of the photon - measurements can thus provide a way to make operations on qubits, although these are not unitary operations.

3 Notation

The notation $\langle v|$ ("bra v") denotes a row vector, the conjugate-transpose of $|v\rangle$, or $|v\rangle^{\dagger}$. For example, $\langle 0| = \left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)$ and $\langle 1| = \left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)$. More generally,

$$\langle\psi| = \left(\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right)^{\dagger} = \left(\begin{smallmatrix}\bar{\alpha} & \bar{\beta}\end{smallmatrix}\right) = \bar{\alpha}\langle 0| + \bar{\beta}\langle 1| \quad . \tag{1}$$

The Dirac notation can be handy. For example, let

$$\left|v_1\right\rangle = a_1\left|0\right\rangle + b_1\left|1\right\rangle, \quad \left|v_2\right\rangle = a_2\left|0\right\rangle + b_2\left|1\right\rangle \quad . \tag{2}$$

Then $\left\langle v_1\middle|v_2\right\rangle$ (shorthand for $\left\langle v_1\right|\left|v_2\right\rangle$) is a matrix product of the 1×2 matrix $\left\langle v_1\right|$ and the 2×1 matrix $\left|v_2\right\rangle$, or just a scalar:

$$\left\langle v_1\middle|v_2\right\rangle = \left(\begin{smallmatrix}\bar{a}_1 & \bar{b}_1\end{smallmatrix}\right)\begin{pmatrix}a_2 \\ b_2\end{pmatrix} = \bar{a}_1 a_2 + \bar{b}_1 b_2 \quad . \tag{3}$$

$\left\langle v_1\middle|v_2\right\rangle = \overline{\left\langle v_2\middle|v_1\right\rangle}$ is an inner product. Note that $\left\langle 0\middle|0\right\rangle = \left\langle 1\middle|1\right\rangle = 1$ and $\left\langle 0\middle|1\right\rangle = \overline{\left\langle 1\middle|0\right\rangle} = 0$. Thus the above equation could have been expanded,

$$\begin{aligned}\left\langle v_1\middle|v_2\right\rangle &= (\bar{a}_1\left\langle 0\right| + \bar{b}_1\left\langle 1\right|)(a_2\left|0\right\rangle + b_2\left|1\right\rangle) \\ &= \bar{a}_1 a_2\left\langle 0\middle|0\right\rangle + \bar{a}_1 b_2\left\langle 0\middle|1\right\rangle + \bar{b}_1 a_2\left\langle 1\middle|0\right\rangle + \bar{b}_1 b_2\left\langle 1\middle|1\right\rangle \\ &= \bar{a}_1 a_2\cdot 1 + \bar{a}_1 b_2\cdot 0 + \bar{b}_1 a_2\cdot 0 + \bar{b}_1 b_2\cdot 1 \\ &= \bar{a}_1 a_2 + \bar{b}_1 b_2 \quad .\end{aligned} \tag{4}$$

In this notation, $\alpha = \left\langle 0\middle|\psi\right\rangle$, $\beta = \left\langle 1\middle|\psi\right\rangle$. The normalization condition $|\alpha|^2 + |\beta|^2 = 1$ is

$$\begin{aligned}1 &= |\alpha|^2 + |\beta|^2 = \bar{\alpha}\alpha + \bar{\beta}\beta \\ &= \left\langle\psi\middle|0\right\rangle\left\langle 0\middle|\psi\right\rangle + \left\langle\psi\middle|1\right\rangle\left\langle 1\middle|\psi\right\rangle \\ &= \left\langle\psi\right|(\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|)\left|\psi\right\rangle \\ &= \left\langle\psi\middle|\psi\right\rangle \quad .\end{aligned} \tag{5}$$

The last equality above follows since $\left|0\right\rangle\left\langle 0\right| = \left(\begin{smallmatrix}1 & 0 \\ 0 & 0\end{smallmatrix}\right)$, $\left|1\right\rangle\left\langle 1\right| = \left(\begin{smallmatrix}0 & 0 \\ 0 & 1\end{smallmatrix}\right)$, so $\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|$ is the 2×2 identity matrix. (This trick is important enough to have its own name, the "resolution of the identity.")

4 Measurement in an arbitrary basis

We may choose any orthogonal basis $v, v^{\perp}$ and measure the qubit in it. To do this, we rewrite our state in that basis: $|\psi\rangle = \alpha'|v\rangle + \beta'|v^{\perp}\rangle$. The outcome is v with probability $|\alpha'|^2$, and $|v^{\perp}\rangle$ with probability $|\beta'|^2$. If the outcome of the measurement on $|\psi\rangle$ yields $|v\rangle$, then as before, the qubit is then in state $|v\rangle$.

We now illustrate this explicitly, using the new notations developed above. We measure $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in the $|v\rangle, |v^{\perp}\rangle$ basis, where $|v\rangle = a|0\rangle + b|1\rangle$. What is the probability of measuring $|v\rangle$?

Our measurement basis vector is $|v\rangle = a|0\rangle + b|1\rangle$. Let's choose the orthogonal complement as $|v^{\perp}\rangle = \overline{b}|0\rangle - \overline{a}|1\rangle$. Check that $\langle v|v^{\perp}\rangle = ab - ba = 0$.

$$
\begin{aligned}
|\psi\rangle &= \Big(|v\rangle\langle v| + |v^{\perp}\rangle\langle v^{\perp}|\Big)|\psi\rangle \\
&= \alpha(|v\rangle\langle v|0\rangle + |v^{\perp}\rangle\langle v^{\perp}|0\rangle) + \beta(|v\rangle\langle v|1\rangle + |v^{\perp}\rangle\langle v^{\perp}|1\rangle) \\
&= (\alpha\langle v|0\rangle + \beta\langle v|1\rangle)|v\rangle + (\alpha\langle v^{\perp}|0\rangle + \beta\langle v^{\perp}|1\rangle)|v^{\perp}\rangle \\
&= (\alpha\overline{a} + \beta\overline{b})|v\rangle + (\alpha b - \beta a)|v^{\perp}\rangle \quad .
\end{aligned}
$$

The probability of measuring $|v\rangle$ in a measurement in the $v, v^{\perp}$ basis is therefore

$$|\langle v|\psi\rangle|^2 = |\alpha\overline{a} + \beta\overline{b}|^2 \quad .$$