1 Readings

Liboff, Introductory Quantum Mechanics, Ch. 11

2 Spin Operators

Last time: $|0\rangle = |\uparrow\rangle = |+\frac{1}{2}\rangle =$ state representing ang. mom. w/ z-comp. up

$|1\rangle = |\downarrow\rangle = |-\frac{1}{2}\rangle =$ state representing ang. mom. w/ z-comp. down

These are the eigenvectors and eigenvalues of the spin for a spin-$\frac{1}{2}$ system, like an electron or proton: $|0\rangle$ and $|1\rangle$ are simultaneous eigenvectors of S^2 and S_z.

$$\begin{aligned}
S^2|0\rangle &= \hbar^2 s(s+1)|0\rangle = \hbar^2\frac{1}{2}(\frac{1}{2}+1)|0\rangle = \frac{3}{4}\hbar^2|0\rangle \\
S^2|1\rangle &= \hbar^2 s(s+1)|1\rangle = \frac{3}{4}\hbar^2|1\rangle \\
S_z|0\rangle &= \hbar m|0\rangle = \frac{1}{2}\hbar|0\rangle, m=+\frac{1}{2} \\
S_z|1\rangle &= \hbar m|1\rangle = -\frac{1}{2}\hbar|0\rangle, m=-\frac{1}{2}
\end{aligned}$$

Results of measurements:
$S^2 \rightarrow \frac{3}{4}\hbar^2, S_z \rightarrow +\frac{\hbar}{2}, -\frac{\hbar}{2}$

Since S_z is a Hamiltonian operator, $|0\rangle$ and $|1\rangle$ form an orthonormal basis that spans the spin-$\frac{1}{2}$ space, which is isomorphic to $\mathscr{C}^2$.

So the most general spin $\frac{1}{2}$ state is $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

Question: How do we represent the spin operators (S^2, S_x, S_y, S_z) in the 2-d basis of the S_z eigenstates $|0\rangle$ and $|1\rangle$?

Answer: They are matrices. Since they act on a two-dimensional vectors space, they must be 2-d matrices. We must calculate their matrix elements:

$$S^2 = \begin{pmatrix} s^2_{11} & s^2_{12} \\ s^2_{21} & s^2_{22} \end{pmatrix}, S_z = \begin{pmatrix} s_{z11} & s_{z12} \\ s_{z21} & s_{z22} \end{pmatrix}, S_x = \begin{pmatrix} s_{x11} & s_{x12} \\ s_{x21} & s_{x22} \end{pmatrix}, \text{ etc. } (S_y)$$

Calculate S^2 matrix: We must sandwich S^2 between all possible combinations of basis vectors. (This is the usual way to construct a matrix!)

$$
\begin{aligned}
s_{11}^2 &= \langle 0|S^2|0\rangle = \langle 0|\frac{3}{4}\hbar^2|0\rangle = \frac{3}{4}\hbar^2 \\
s_{12}^2 &= \langle 0|S^2|1\rangle = \langle 0|\frac{3}{4}\hbar^2|1\rangle = 0 \\
s_{21}^2 &= \langle 1|S^2|0\rangle = \langle 1|\frac{3}{4}\hbar^2|0\rangle = 0 \\
s_{22}^2 &= \langle 1|S^2|1\rangle = \langle 1|\frac{3}{4}\hbar^2|1\rangle = \frac{3}{4}\hbar^2
\end{aligned}
$$

So $S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Find the S_z matrix:

$$
\begin{aligned}
s_{z11}^2 &= \langle 0|S_z|0\rangle = \langle 0|+\frac{\hbar}{2}|0\rangle = \frac{\hbar}{2} \\
s_{z12}^2 &= \langle 0|S_z|1\rangle = \langle 0|-\frac{\hbar}{2}|1\rangle = 0 \\
s_{z21}^2 &= \langle 1|S_z|0\rangle = \langle 1|+\frac{\hbar}{2}|0\rangle = 0 \\
s_{z22}^2 &= \langle 1|S_z|1\rangle = \langle 1|-\frac{\hbar}{2}|1\rangle = -\frac{\hbar}{2}
\end{aligned}
$$

So $S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Find S_x matrix: This is more difficult

What is $S_{x11} = \langle 0|S_x|0\rangle$? $|0\rangle$ is not an eigenstate of S_z, so it's not trivial.

Use raising and lowering operators: $S_\pm = S_x \pm iS_y$

$\Rightarrow S_x = \frac{1}{2}(S_+ + S_-), S_y = \frac{1}{2i}(S_+ - S_-)$

$\Rightarrow S_{x11} = \langle 0|\frac{1}{2}(S_+ + S_-)|0\rangle \Rightarrow S_+|0\rangle = 0$, since $|0\rangle$ is the highest S_z state.

But what is $S_-|0\rangle$? Since S_- is the lowering operator, we know that $S_-|0\rangle \propto |1\rangle$. That is $S_-|0\rangle = A_-|1\rangle$ for some complex number A_- which we have yet to determine. Similarly, $S_+|1\rangle = A_+|0\rangle$.

Question: What is A_-?

Answer:

$$
\begin{aligned}
A_+ &= \hbar\sqrt{s(s+1)-m(m+1)} \to S_+|s,m\rangle = A_+|s,m+1\rangle \\
A_- &= \hbar\sqrt{s(s+1)-m(m-1)} \to S_-|s,m\rangle = A_-|s,m-1\rangle
\end{aligned}
$$

Proof: First note that $|A_-|^2 = \langle s,m|S_+S_-|s,m\rangle$.

Since $S_\pm = S_x \pm iS_y$, where S_x, S_y are Hermitian, $S_+^\dagger = S_-$. Hence $A_+(s,m) = A_-(s,m+1)^*$. We are free to choose a phase for the eigenstates; set it so $A_+ \geq 0$ (real and nonnegative). Now

$$\begin{aligned}\langle s,m|\,S_-S_+|s,m\rangle &= (S_-^\dagger|s,m\rangle)^\dagger S_+|s,m\rangle = (S_+|s,m\rangle)^\dagger S_+|s,m\rangle \\ &= A_+^2\langle s,m|\,s,m\rangle = A_+^2 \ .\end{aligned}$$

Also,

$$\begin{aligned}S_-S_+ &= (S_x - iS_y)(S_x + iS_y) = S_x^2 + S_y^2 + i[S_x,S_y] \\ &= S^2 - S_z^2 - \hbar S_z \ .\end{aligned}$$

Therefore,

$$\begin{aligned}\langle s,m|\,S_-S_+|s,m\rangle &= \langle s,m|\,(S^2 - S_z^2 - \hbar S_z)|s,m\rangle \\ &= \hbar^2 s(s+1) - (\hbar m)^2 - \hbar(\hbar m) \\ &= \hbar^2(s(s+1) - m(m+1)) \ ,\end{aligned}$$

using $S^2|s,m\rangle = \hbar^2 s(s+1)|s,m\rangle$ and $S_z|s,m\rangle = \hbar m|s,m\rangle$. Thus

$$\begin{aligned}A_+(s,m) &= \hbar\sqrt{s(s+1) - m(m+1)} \\ A_-(s,m) &= A_+(s,m-1) = \hbar\sqrt{s(s+1) - m(m-1)} \ .\end{aligned}$$

Now we use these values $A_\pm$ to find the desired coefficients:

$$\begin{aligned}S_+|0\rangle &= 0 \\ S_+|1\rangle &= \hbar\sqrt{\frac{1}{2}(\frac{1}{2}+1) - (-\frac{1}{2})(-\frac{1}{2}+1)}|0\rangle = \hbar|0\rangle \\ S_-|0\rangle &= \hbar\sqrt{\frac{1}{2}(\frac{1}{2}+1) - (\frac{1}{2})(\frac{1}{2}-1)}|1\rangle = \hbar|1\rangle \\ S_-|1\rangle &= 0\end{aligned}$$

$\Rightarrow S_{x11} = \frac{1}{2}\langle 0|\,(S_+ + S_-)|0\rangle = \frac{1}{2}\langle 0|\,[S_+|0\rangle + S_-|0\rangle]$

$$\begin{aligned}S_{x11} &= \frac{1}{2}\langle 0|\,[0 + \hbar|1\rangle] = 0 \\ S_{x12} &= \langle 0|\,\frac{1}{2}(S_+ + S_-)|1\rangle = \frac{1}{2}\langle 0|\,[\hbar|0\rangle + 0] = \frac{\hbar}{2} \\ S_{x21} &= \langle 1|\,\frac{1}{2}(S_+ + S_-)|0\rangle = \frac{1}{2}\langle 1|\,[0 + \hbar|1\rangle] = \frac{\hbar}{2} \\ S_{x22} &= \langle 1|\,\frac{1}{2}(S_+ + S_-)|1\rangle = \frac{1}{2}\langle 1|\,[\hbar|0\rangle + 0] = 0\end{aligned}$$

So $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Find S_y matrix: Use $S_y = \frac{1}{2i}(S_+ - S_-)$. The proceed similarly to above for S_x. Check it out yourself to see that you understand the matrix and bra-ket mechanics.

Answer: $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

In summary, we define

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

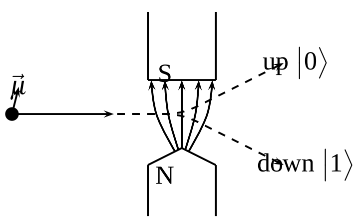
Then, $S^2 = \frac{3}{4}\hbar^2\sigma_0,\ S_x = \frac{\hbar}{2}\sigma_1,\ S_y = \frac{\hbar}{2}\sigma_2,\ S_z = \frac{\hbar}{2}\sigma_3$

$\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are called the Pauli Spin Matrices. They are very important for understanding the behavior of two-level systems. Note that we have already encountered these in our discussion of qubits, where they correspond to the gates $X = \sigma_1,\ Y = \sigma_2,\ Z = \sigma_3,\ I = \sigma_0$. In the next couple of lectures we shall frequently interconvert, using

$$S_x = \frac{\hbar}{2}X,$$
$$S_y = \frac{\hbar}{2}Y,$$
$$S_z = \frac{\hbar}{2}Z.$$

3 Measuring Spin

We can measure the spin state m with a Stern-Gerlach device. This is simply a magnet set up to generate a particular inhomogeneous $\vec{B}$ field. In the figure below, the field is strong near the N pole (below) and weaker near the S pole (above). If the z-axis points upwards, then we have a negative field gradient in the z direction, i.e., $\partial B/\partial z < 0$.



When a particle with spin state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is shot through the apparatus from the left, its spin-up portion is deflected upward, and its spin-down portion downward. The particle's spin becomes entangled with its position! Placing detectors to intercept the outgoing paths therefore measures the particle's spin.

Why does this work? We'll give a semiclassical explanation – mixing the classical $\vec{F} = m\vec{a}$ and the quantum $H\psi = E\psi$ – which is not really correct, but gives the correct intuition. [See Griffith's § 4.4.2, pp. 162-164 for a more complete argument.] Now the potential energy due to the spin interacting with the field is

$$E = -\vec{\mu}\cdot\vec{B} \ ,$$

so the associated force is

$$\vec{F}_{\text{spin}} = -\vec{\nabla}E = \vec{\nabla}(\vec{\mu}\cdot\vec{B}) \ .$$

At the center $\vec{B} = B(z)\hat{z}$, with $\frac{\partial B}{\partial z} < 0$, so $\vec{F} = \vec{\nabla}(\mu_z B(z)) = \mu_z \frac{\partial B}{\partial z}\hat{z}$. The magnetic moment $\vec{\mu}$ is related to spin $\vec{S}$ by $\vec{\mu} = \frac{gq}{2m}\vec{S} = -\frac{e}{m}\vec{S}$ for an electron. Hence

$$\vec{F} = \frac{e}{m}|\frac{\partial B}{\partial z}|S_z\hat{z} \ ;$$

if the electron is spin up ($S_z = +1/2$), the force is upward, and if the electron is spin down ($S_z = -1/2$), the force is downward.

4 Initialize a Spin Qubit

- How can we create a beam of qubits in the state $|\psi\rangle = |0\rangle$? Pass a beam of spin-$\frac{1}{2}$ particles with randomly oriented spins through a Stern-Gerlach apparatus oriented along the z axis. Block the downward-pointing beam, leaving the other beam of $|0\rangle$ qubits.

  Note that we *measure* the spin when we intercept an outgoing beam – measurement process is probabilistic and not unitary.

- How can we create a beam of qubits in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$? First find the point on the Bloch sphere corresponding to $|\psi\rangle$. That is, write

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

  (up to a phase), where

$$\tan\frac{\theta}{2} = |\frac{\beta}{\alpha}| \qquad e^{i\varphi} = \frac{\beta/|\beta|}{\alpha/|\alpha|} \ .$$

  The polar coordinates θ, φ determine a unit vector $\hat{n} = \cos\varphi\sin\theta\hat{x} + \sin\varphi\sin\theta\hat{y} + \cos\theta\hat{z}$. Now just point the Stern-Gerlach device in the corresponding direction on the Bloch sphere, and intercept one of the two outgoing beams. That is, a Stern-Gerlach device pointed in direction $\hat{n}$ measures $S_{\hat{n}} = \hat{n}\cdot\hat{\vec{S}}$.

  Some details for this. We need to find the eigenstates of $S_n = \vec{S}\cdot\hat{n}$. First express this in cartesian coordinates, using $\hat{n} = \sin\theta\cos\phi\hat{x} + \sin\theta\cos\phi\hat{y} + \cos\theta\hat{z}$, and the relations $S_\alpha = \hbar/2\alpha, \alpha = X, Y, Z$. We thereby arrive at the 2x2 matrix representation of S_n in the z-basis:

$$S_n = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & \cos\theta \end{pmatrix},$$

  Now diagonalize this to obtain eigenvalues $\pm\hbar/2$ (why are you not surprised?) and eigenstates

$$\begin{aligned} |0\rangle_n &= \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \ \ (s_n = +\frac{\hbar}{2}) \\ |1\rangle_n &= -e^{-i\theta}\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle \ \ (s_n = -\frac{\hbar}{2}). \end{aligned}$$

  The first eigenstate is the desired general state on the Bloch sphere, so we just need to intercept the positive spin eigenstate, i.e., upward deflected beam in the S-G frame.

- Can we use an S-G apparatus to implement a unitary (deterministic) transformation? No, since the S-G apparatus makes a measurement and is not unitary. In other words, it collapses the wave function to one component only. It is fine for initializing an arbitrary state, but to make a unitary transformation we have to evolve the wave function according to a Hamiltonian $\hat{H}$:

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle$$

This rotates the spin qubit wave function on the Bloch sphere. In the next lecture we will see how to accomplish an arbitrary single-qubit unitary gate (a arbitrary rotation on the Bloch sphere) by two different approaches, Larmor precession and spin resonance.