# CS-184: Computer Graphics

Lecture #13: Natural Splines, B-Splines, and NURBS

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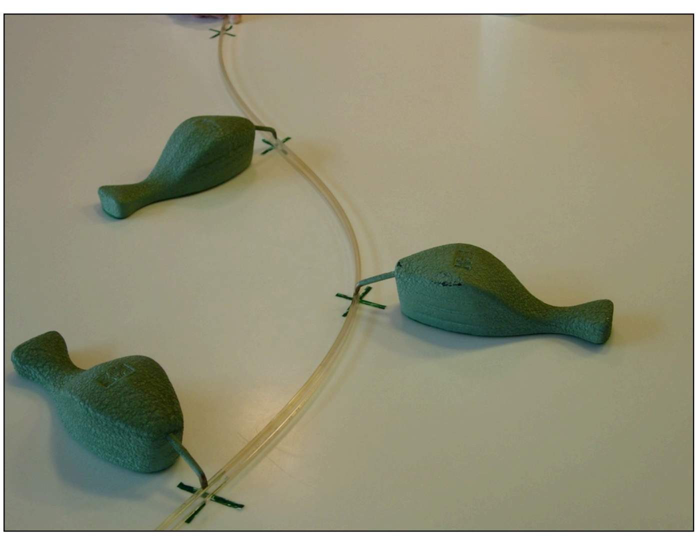
# Natural Splines

- Draw a "smooth" line through several points



A real draftsman's spline.

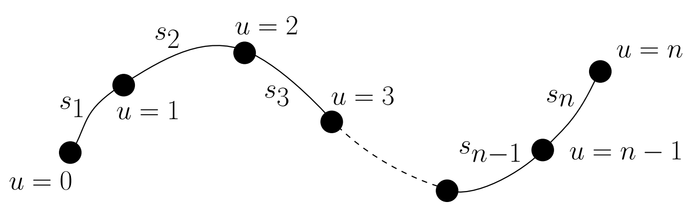
Image from Carl de Boor's webpage.

# Natural Cubic Splines

- Given $n+1$ points
  - Generate a curve with $n$ segments
  - Curves passes through points
  - Curve is $C^2$ continuous
- Use cubics because lower order is better...

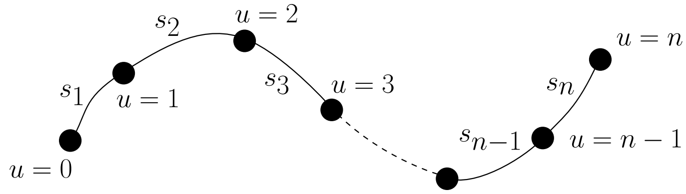


# Natural Cubic Splines




$$\mathbf{x}(u) = \begin{cases} \mathbf{s}_1(u) & \text{if } 0 \leq u < 1 \\ \mathbf{s}_2(u-1) & \text{if } 1 \leq u < 2 \\ \mathbf{s}_3(u-2) & \text{if } 2 \leq u < 3 \\ & \vdots \\ \mathbf{s}_n(u-(n-1)) & \text{if } n-1 \leq u \leq n \end{cases}$$

# Natural Cubic Splines




$s_i(0) = p_{i-1} \quad i = 1 \ldots n$ ← $n$ constraints

$s_i(1) = p_i \quad i = 1 \ldots n$ ← $n$ constraints

$s'_i(1) = s'_{i+1}(0) \quad i = 1 \ldots n - 1$ ← $n$-1 constraints

$s''_i(1) = s''_{i+1}(0) \quad i = 1 \ldots n - 1$ ← $n$-1 constraints

$s''_1(0) = s''_n(1) = 0$ ←2 constraints

**Total 4$n$ constraints**



# Natural Cubic Splines

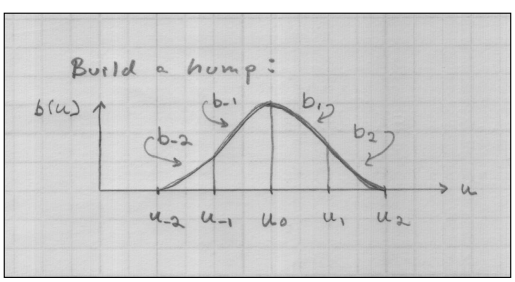
- Interpolate data points
- No convex hull property
- Non-local support
  - Consider matrix structure...
- $C^2$ using cubic polynomials

# B-Splines

- Goal: $C^2$ cubic curves with local support
  - Give up interpolation
  - Get convex hull property
- Build basis by designing "hump" functions



# B-Splines



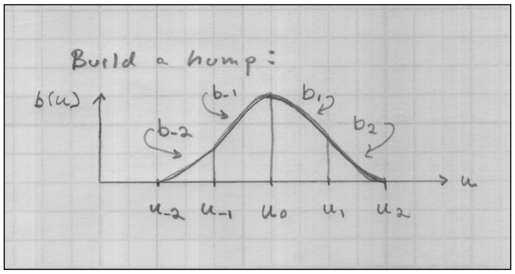

$$\mathbf{b}(u) = \begin{cases} \mathbf{b_{-2}}(u) & \text{if } u_{-2} \leq u < u_{-1} \\ \mathbf{b_{-1}}(u) & \text{if } u_{-1} \leq u < u_0 \\ \mathbf{b_{+1}}(u) & \text{if } u_0 \leq u < u_{+1} \\ \mathbf{b_{+2}}(u) & \text{if } u_{+1} \leq u \leq u_{+2} \end{cases}$$

$b''_{-2}(u_{-2}) = b'_{-2}(u_{-2}) = b_{-2}(u_{-2}) = 0$ ←3 constraints
$b''_{+2}(u_{+2}) = b'_{+2}(u_{+2}) = b_{+2}(u_{+2}) = 0$ ←3 constraints

$b_{-2}(u_{-1}) = b_{-1}(u_{-1})$
$b_{-1}(u_0) = b_{+1}(u_0)$
$b_{+1}(u_{+1}) = b_{+2}(u_{+1})$
← [ Repeat for $b'$ and $b''$
3×3=9 constraints ]

**Total 15 constraints** ...... need one more

# B-Splines




$$\mathbf{b}(u) = \begin{cases} \mathbf{b_{-2}}(u) & \text{if } u_{-2} \leq u < u_{-1} \\ \mathbf{b_{-1}}(u) & \text{if } u_{-1} \leq u < u_{0} \\ \mathbf{b_{+1}}(u) & \text{if } u_{0} \leq u < u_{+1} \\ \mathbf{b_{+2}}(u) & \text{if } u_{+1} \leq u \leq u_{+2} \end{cases}$$

$b''_{-2}(u_{-2}) = b'_{-2}(u_{-2}) = b_{-2}(u_{-2}) = 0$ ←3 constraints

$b''_{+2}(u_{+2}) = b'_{+2}(u_{+2}) = b_{+2}(u_{+2}) = 0$ ←3 constraints

$b_{-2}(u_{-1}) = b_{-1}(u_{-1})$

$b_{-1}(u_{0}) = b_{+1}(u_{0})$

$b_{+1}(u_{+1}) = b_{+2}(u_{+1})$

← Repeat for $b'$ and $b''$; 3×3=9 constraints

$b_{-2}(u_{-2}) + b_{-1}(u_{-1}) + b_{+1}(u_{0}) + b_{+2}(u_{+1}) = 1$ ←1 constraint (convex hull)

**Total 16 constraints**



# B-Splines

- Build a curve w/ overlapping bumps
- Continuity
  - Inside bumps $C^2$
  - Bumps "fade out" with $C^2$ continuity
- Boundaries
  - Circular
  - Repeat end points
  - Extra end points

# B-Splines

- Notation
  - The basis functions are the $b_i(u)$
  - "Hump" functions are the concatenated function
    - Sometimes the humps are called basis... can be confusing
  - The $u_i$ are the knot locations
  - The weights on the hump/basis functions are control points



# B-Splines

- Similar construction method can give higher continuity with higher degree polynomials
- Repeating knots drops continuity
  - Limit as knots approach each other
- Still cubics, so conversion to other cubic basis is just a matrix multiplication

# B-Splines

- Geometric construction
  - Due to Cox and de Boor
  - My own notation, beware if you compare w/ text

- Let hump centered on $u_i$ be $N_{i,4}(u)$

Cubic is order 4

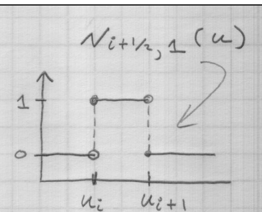
$N_{i,k}(u)$ Is order $k$ hump, centered at $u_i$

Note: $i$ is integer if $k$ is even
else $(i+1/2)$ is integer



# B-Splines

$$N_{i,1}(u) = \begin{cases} 1 & \text{if } u_{i-1/2} \le u < u_{i+1/2} \\ 0 & \text{else} \end{cases}$$



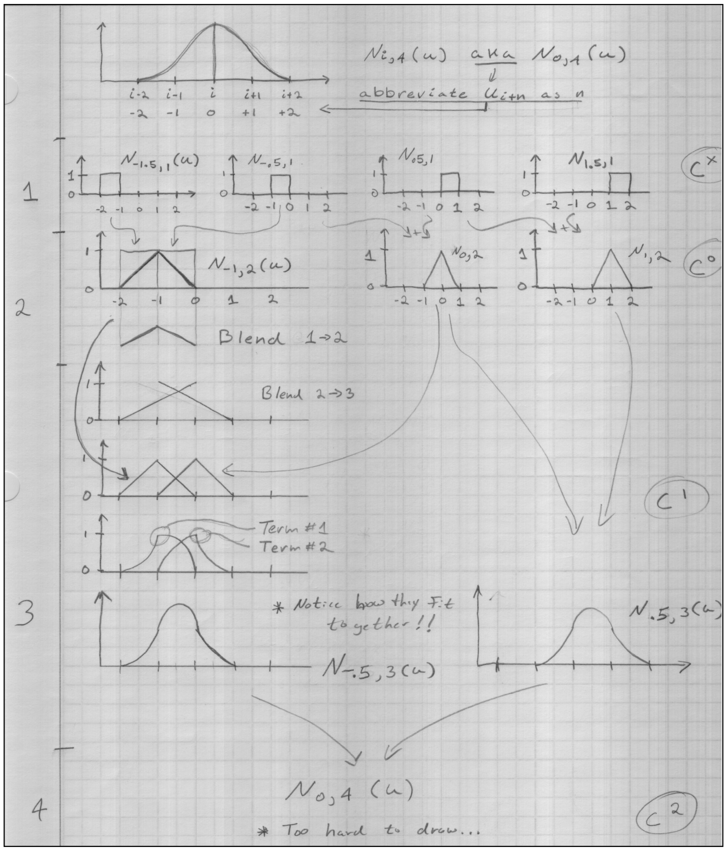

$$N_{i,k}(u) = \frac{(u - u_{i-k/2})\, N_{i-1/2,k-1}(u)}{u_{i+k/2-1} - u_{i-k/2}} \quad \leftarrow \text{“Term \#1”}$$

$k \ge 2$

$$+$$

$$\frac{(u_{i+k/2} - u)\, N_{i+1/2,k-1}(u)}{u_{i+k/2} - u_{i-k/2+1}} \quad \leftarrow \text{“Term \#2”}$$

recursive defn.






# NURBS

- **N**on**u**niform **R**ational **B-S**plines
  - Basically B-Splines using homogeneous coordinates
  - Transform under perspective projection
  - A bit of extra control

# NUBRS

$$\mathbf{p}_i = \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \\ p_{iw} \end{bmatrix}$$

$$\mathbf{x}(u) = \frac{\sum_i \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \end{bmatrix} N_i(u)}{\sum_i p_{iw} N_i(u)}$$

- Non-linear in the control points
- The $p_{iw}$ are sometimes called "weights"