CS-184: Computer Graphics

Lecture #12: Curves and Surfaces

Prof. James O'Brien
University of California, Berkeley

V2007-F-12-1.0

Today

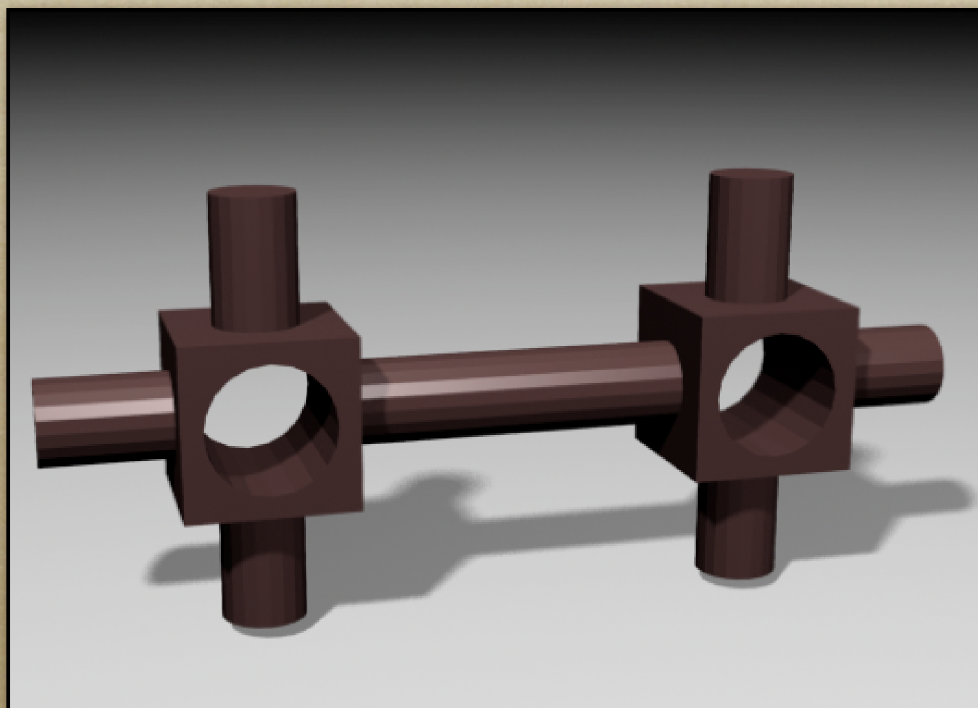
- General curve and surface representations
- Splines and other polynomial bases

Geometry Representations

- Constructive Solid Geometry (CSG)
- Parametric
  - Polygons
  - Subdivision surfaces
- Implicit Surfaces
- Point-based Surface

- Not always clear distinctions
  - *i.e.* CSG done with implicits

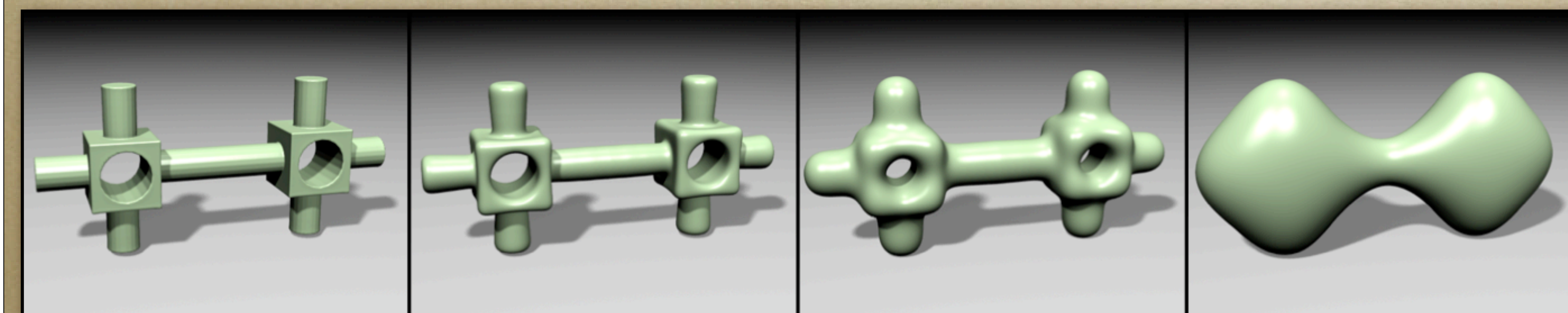
Geometry Representations



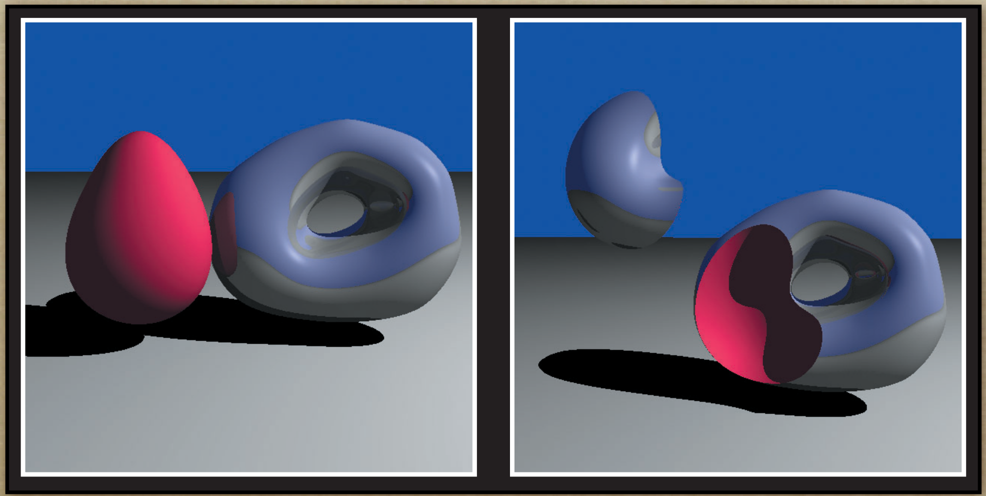
Object made by CSG
Converted to polygons

Geometry Representations

Object made by CSG
Converted to polygons
Converted to implicit surface

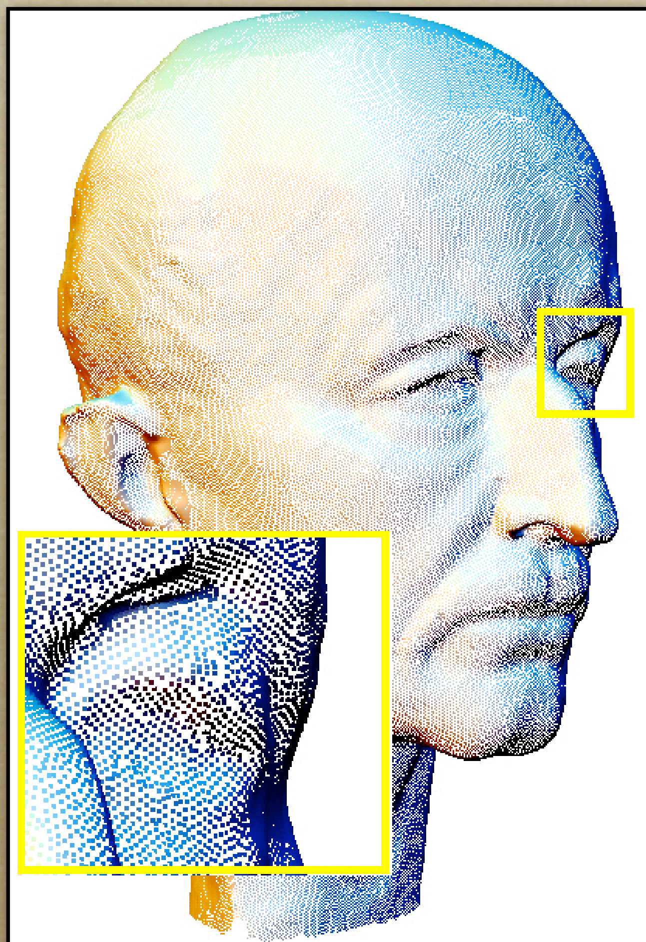


Geometry Representations



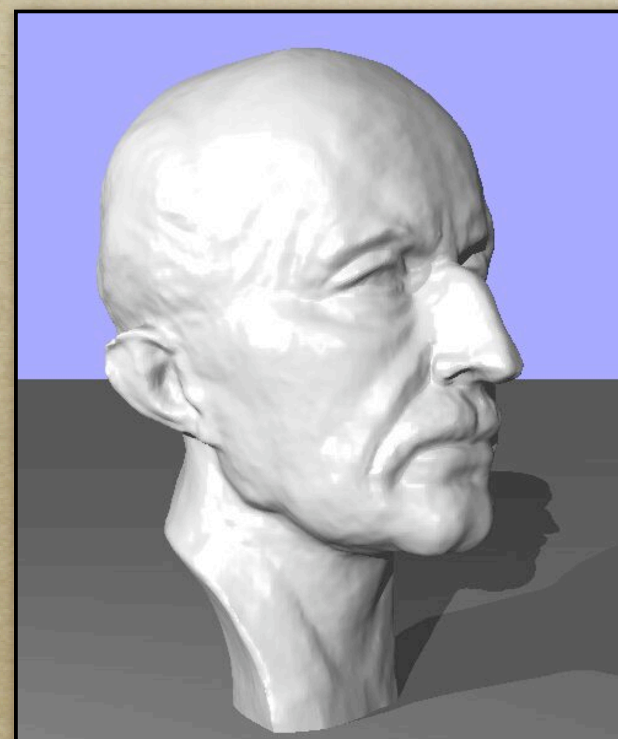
CSG on implicit surfaces

Geometry Representations

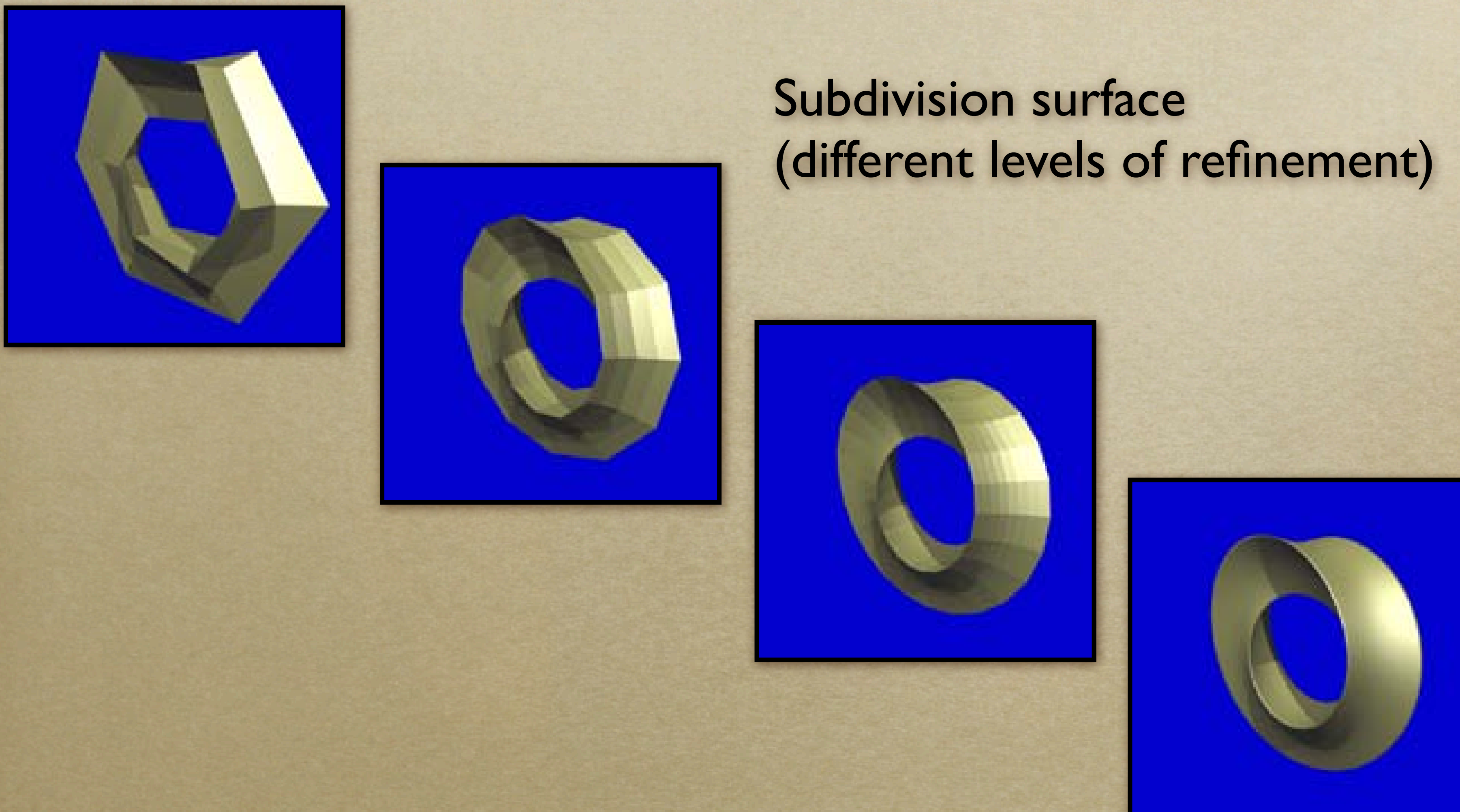


Ohtake, *et al.*, SIGGRAPH 2003

Point-based surface descriptions



Geometry Representations



Subdivision surface
(different levels of refinement)

Images from Subdivision.org

Geometry Representations

- Various strengths and weaknesses
  - Ease of use for design
  - Ease/speed for rendering
  - Simplicity
  - Smoothness
  - Collision detection
  - Flexibility (in more than one sense)
  - Suitability for simulation
  - *many others...*

Parametric Representations

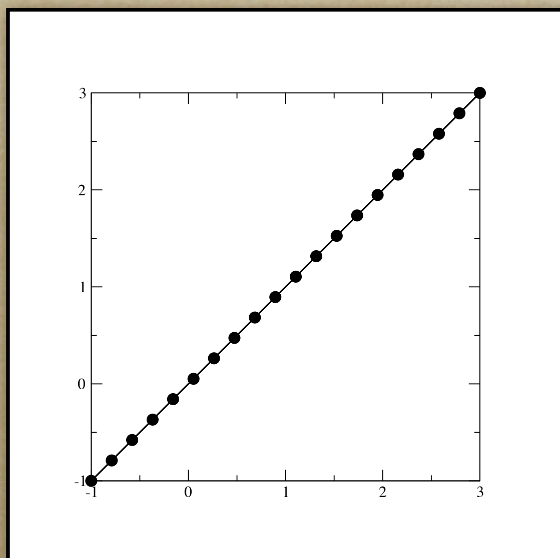
Curves:	$\boldsymbol{x} = \boldsymbol{x}(u)$	$\boldsymbol{x} \in \Re^n$	$u \in \Re$
Surfaces:	$\boldsymbol{x} = \boldsymbol{x}(u, v)$	$\boldsymbol{x} \in \Re^n$	$u, v \in \Re$
	$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{u})$		$\boldsymbol{u} \in \Re^2$
Volumes:	$\boldsymbol{x} = \boldsymbol{x}(u, v, w)$	$\boldsymbol{x} \in \Re^n$	$u, v, w \in \Re$
	$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{u})$		$\boldsymbol{u} \in \Re^3$

and so on...

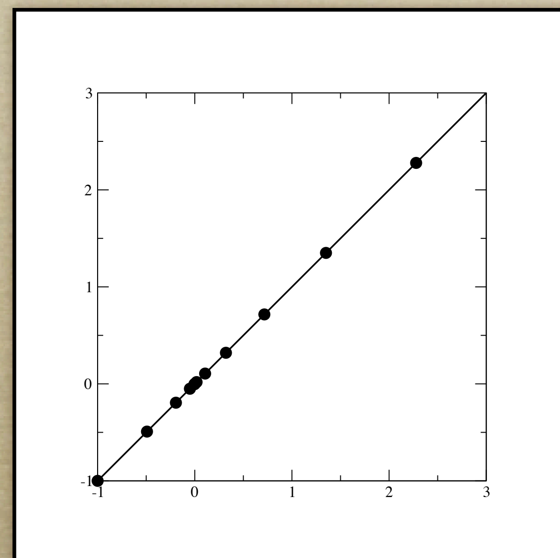
Note: a vector function is really n scalar functions

Parametric Rep. Non-unique

- Same curve/surface may have multiple formulae



$\boldsymbol{x}(u) = [u, u]$

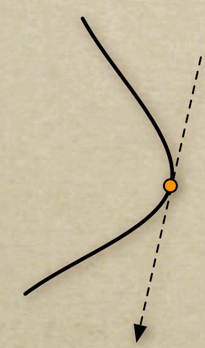


$\boldsymbol{x}(u) = [u^3, u^3]$

Simple Differential Geometry

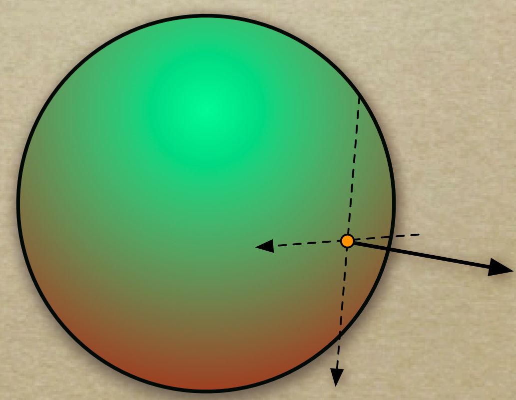
- Tangent to curve

$$\boldsymbol{t}(u) = \left.\frac{\partial \boldsymbol{x}}{\partial u}\right|_u$$



- Tangents to surface

$$\boldsymbol{t}_u(u,v) = \left.\frac{\partial \boldsymbol{x}}{\partial u}\right|_{u,v} \qquad \boldsymbol{t}_v(u,v) = \left.\frac{\partial \boldsymbol{x}}{\partial v}\right|_{u,v}$$



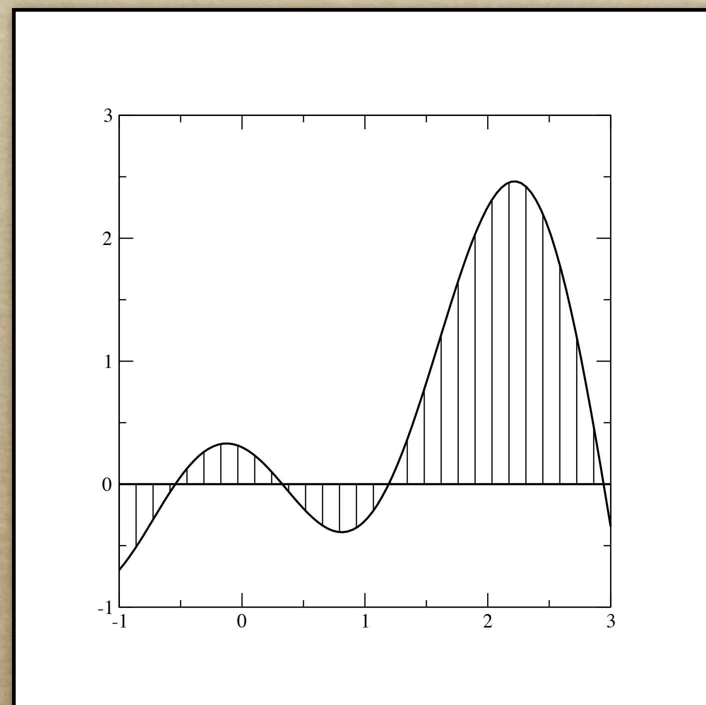
- Normal of surface

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{t}_u \times \boldsymbol{t}_v}{||\boldsymbol{t}_u \times \boldsymbol{t}_v||}$$

- Also: curvature, curve normals, curve bi-normal, *others*...
- Degeneracies: $\partial \boldsymbol{x}/\partial u = 0$ or $\boldsymbol{t}_u \times \boldsymbol{t}_v = 0$

Discretization

- Arbitrary curves have an uncountable number of parameters



i.e. specify function value at all points on real number line

Discretization

- Arbitrary curves have an uncountable number of parameters
- Pick *complete* set of basis functions
  - Polynomials, Fourier series, *etc*.

$$x(u) = \sum_{i=0}^{\infty} c_i \phi_i(u)$$

- Truncate set at some reasonable point

$$x(u) = \sum_{i=0}^{3} c_i \phi_i(u) = \sum_{i=0}^{3} c_i \, u^i$$

- Function represented by the vector (list) of c_i
- The c_i may themselves be vectors

$$\boldsymbol{x}(u) = \sum_{i=0}^{3} \boldsymbol{c}_i \phi_i(u)$$

Polynomial Basis

- Power Basis

$$x(u) = \sum_{i=0}^{d} c_i\, u^i$$

$$x(u) = \boldsymbol{C} \cdot \boldsymbol{\mathcal{P}}^d \qquad \begin{aligned} \boldsymbol{C} &= [c_0, c_1, c_2, \ldots, c_d] \\ \boldsymbol{\mathcal{P}}^d &= [1, u, u^2, \ldots, u^d] \end{aligned}$$

The elements of $\mathcal{P}^d$ are *linearly independant*

i.e. no good approximation

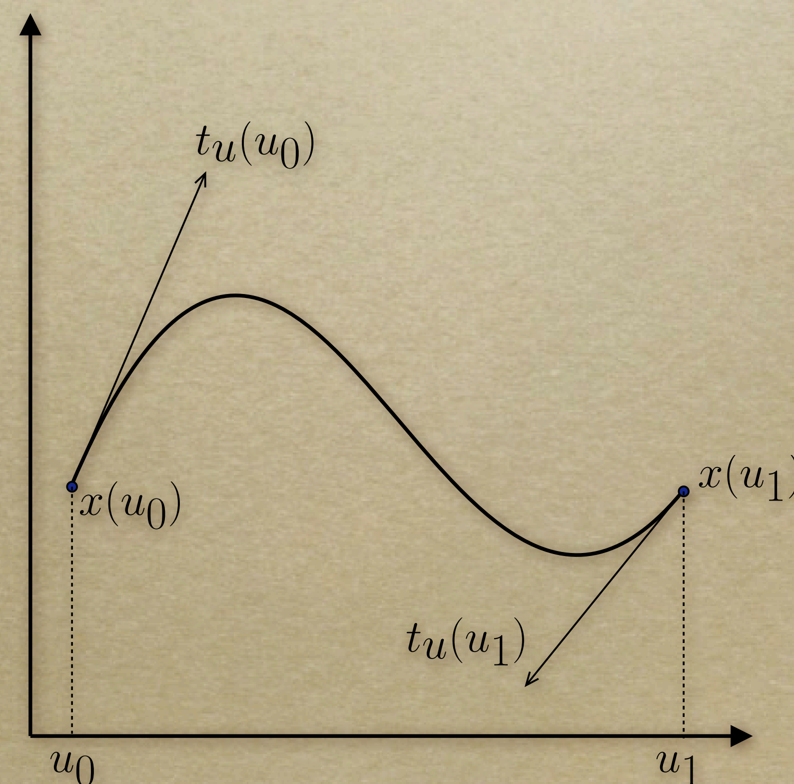
$$u^k \not\approx \sum_{i \neq k} c_i\, u^i$$

Skipping something would lead to bad results... odd stiffness

Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

For now, assume
$u_0 = 0 \quad u_1 = 1$



Specifying a Curve

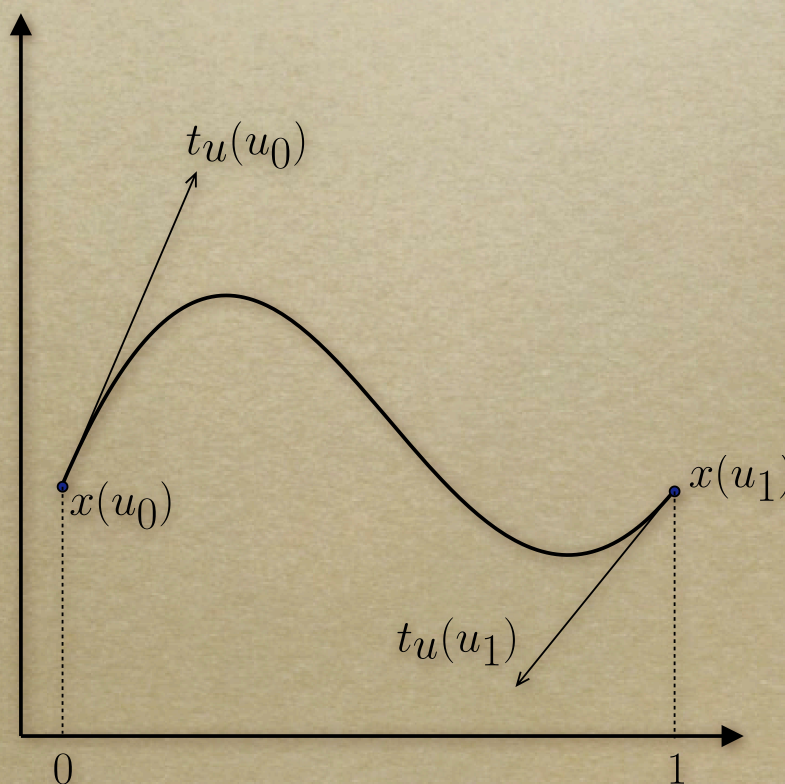
Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$x(0) = c_0 \quad = x_0$$

$$x(1) = \Sigma\, c_i \ = x_1$$

$$x'(0) = c_1 \quad = x'_0$$

$$x'(1) = \Sigma\, i\, c_i = x'_1$$

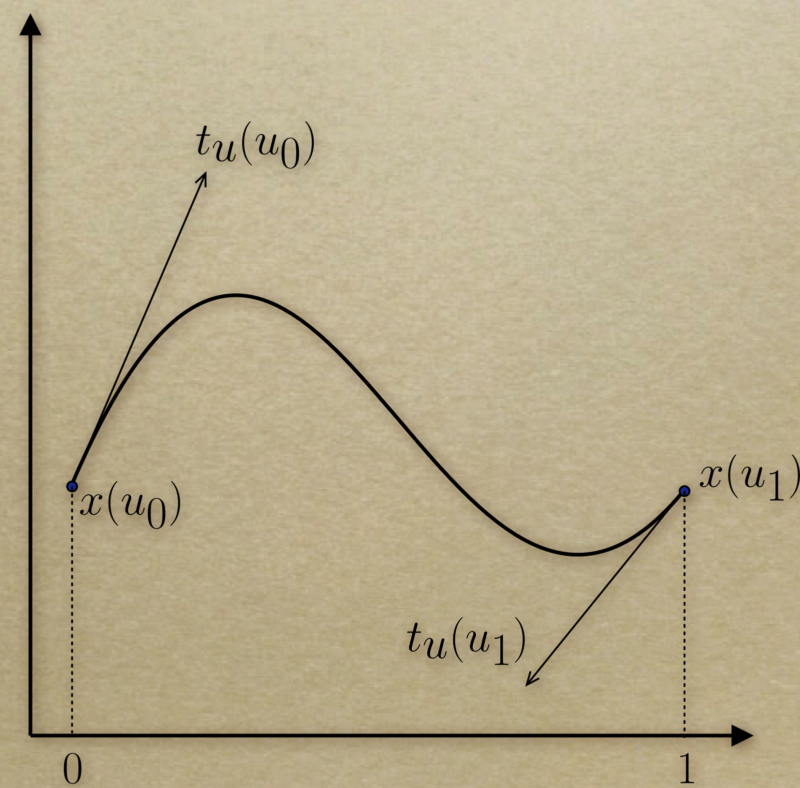


Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

$$\begin{bmatrix} x_0 \\ x_1 \\ x'_0 \\ x'_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\mathbf{p} = \mathbf{B} \cdot \mathbf{c}$$

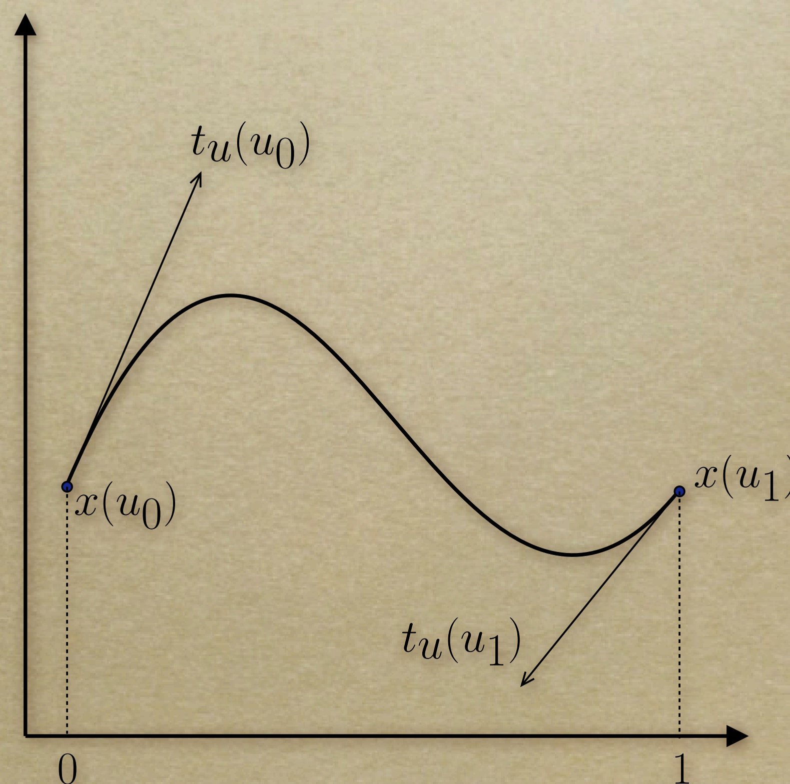


Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?

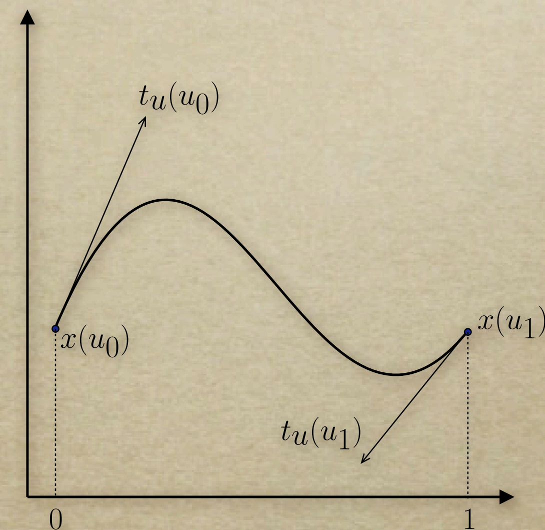
$$\mathbf{c} = \beta_{\mathrm{H}} \cdot \mathbf{p}$$

$$\boldsymbol{\beta}_{\mathrm{H}} = \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & 1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$



Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?



$$\mathbf{c} = \beta_{\mathrm{H}} \cdot \mathbf{p}$$

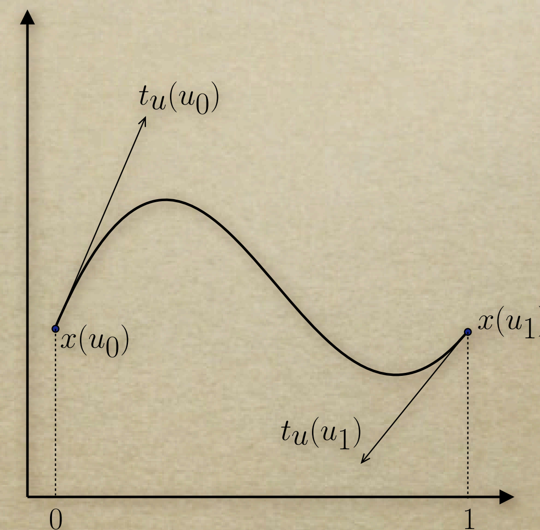
$$x(u) = \mathcal{P}^3 \cdot \mathbf{c} = \boxed{\mathcal{P}^3 \boldsymbol{\beta}_{\mathrm{H}}}\mathbf{p}$$

$$= \begin{bmatrix} 1 + 0u - 3u^2 + 2u^3 \\ 0 + 0u + 3u^2 - 2u^3 \\ 0 + 1u - 2u^2 + 1u^3 \\ 0 + 0u - 1u^2 + 1u^3 \end{bmatrix} \mathbf{p}$$

$$\boldsymbol{\beta}_{\mathrm{H}} = \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & 1 \\ 2 & -2 & 1 & 1 \end{bmatrix}$$

Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?



$$\mathbf{c} = \beta_{\mathrm{H}} \cdot \mathbf{p}$$

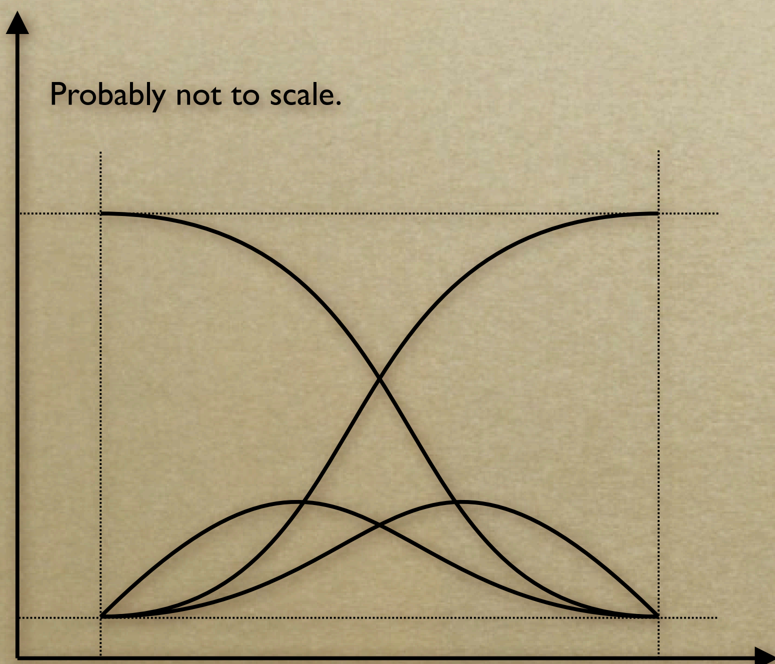
$$x(u) = \begin{bmatrix} 1 + 0u - 3u^2 + 2u^3 \\ 0 + 0u + 3u^2 - 2u^3 \\ 0 + 1u - 2u^2 + 1u^3 \\ 0 + 0u - 1u^2 + 1u^3 \end{bmatrix} \mathbf{p}$$

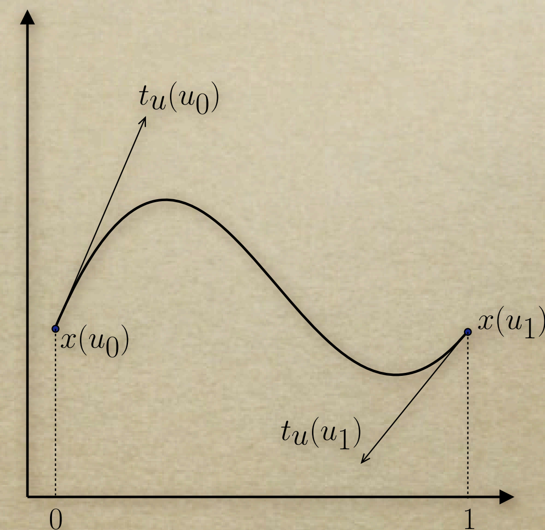
$$x(u) = \sum_{i=0}^{3} p_i b_i(u)$$

Hermite basis functions

Specifying a Curve

Given desired values (constraints) how do we determine the coefficients for cubic power basis?





$$x(u) = \sum_{i=0}^{3} p_i b_i(u)$$

Hermite basis functions

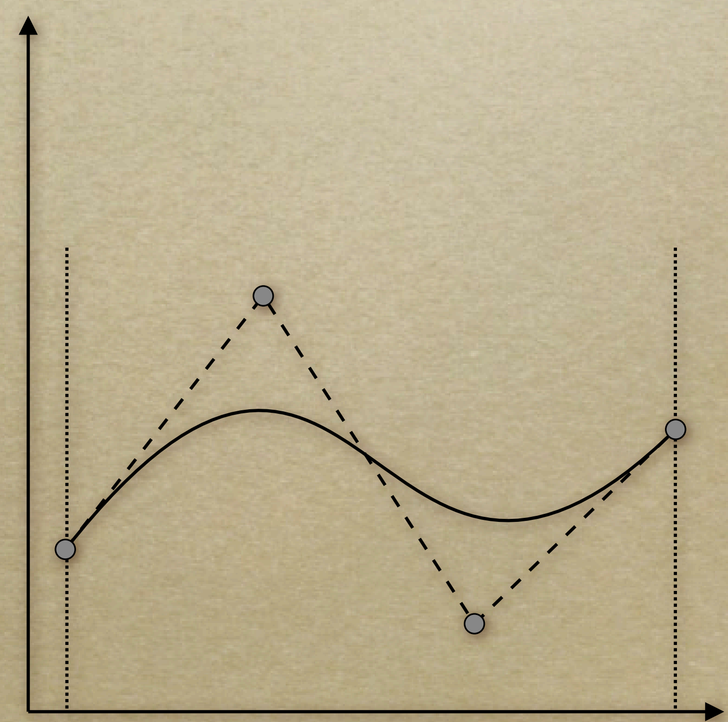
Hermite Basis

- Specify curve by
  - Endpoint values
  - Endpoint tangents (derivatives)
- Parameter interval is arbitrary (most times)
  - Don't need to recompute basis functions
- These are *cubic* Hermite
  - Could do construction for any odd degree
  - $(d-1)/2$ derivatives at end points

Cubic Bézier

- Similar to Hermite, but specify tangents indirectly

$$x_0 = p_0$$
$$x_1 = p_3$$
$$x_0' = 3(p_1 - p_0)$$
$$x_1' = 3(p_3 - p_2)$$



Note: all the control points are points in space, no tangents.

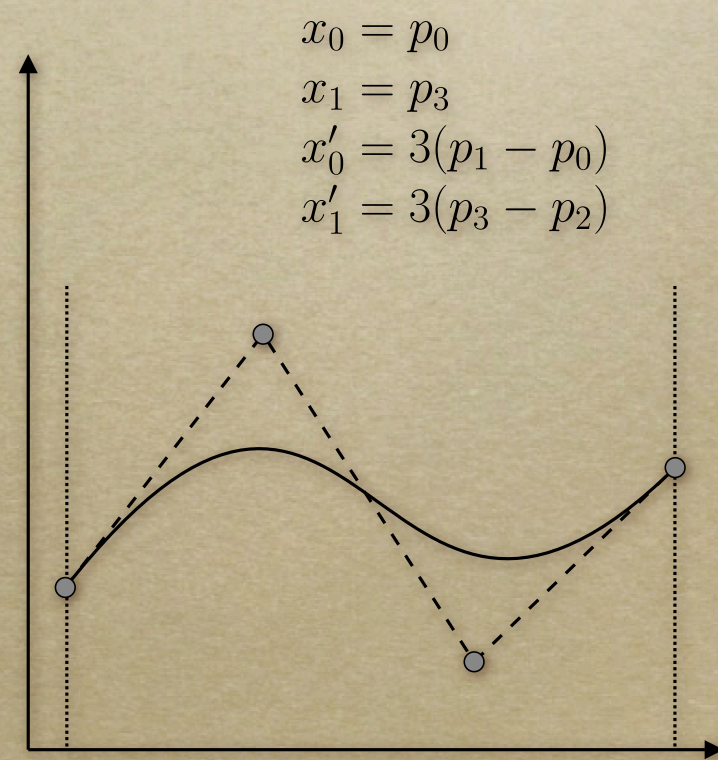
Cubic Bézier

- Similar to Hermite, but specify tangents indirectly

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \mathbf{p}$$

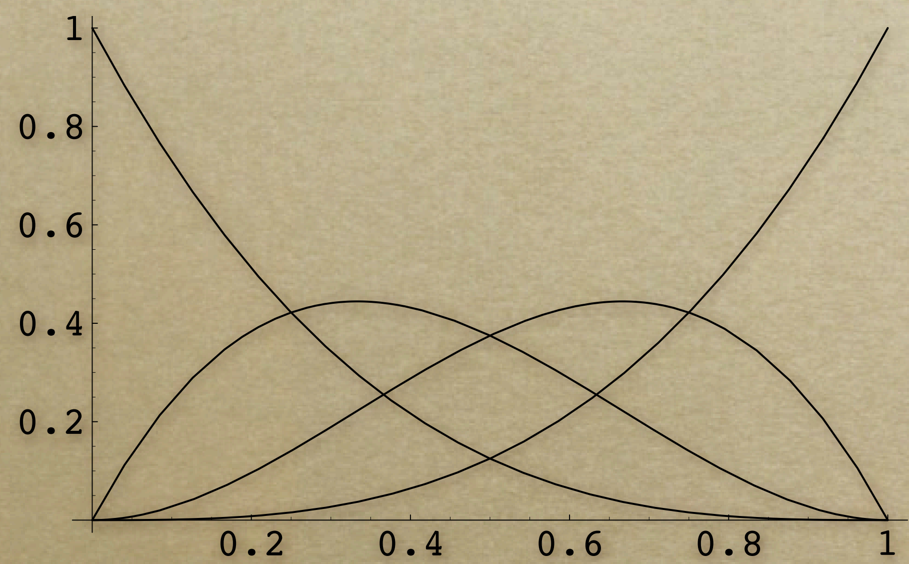
$$\mathbf{c} = \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}} \mathbf{p}$$

$$\mathbf{c} = \boldsymbol{\beta}_{\mathrm{Z}}\,\mathbf{p}$$



Cubic Bézier

- Plot of Bézier basis functions



Changing Bases

- Power basis, Hermite, and Bézier all are still just cubic polynomials
  - The three basis sets all span the same space
  - Like different axes in $\Re^{\cancel{3}}$ $\Re^4$
- Changing basis

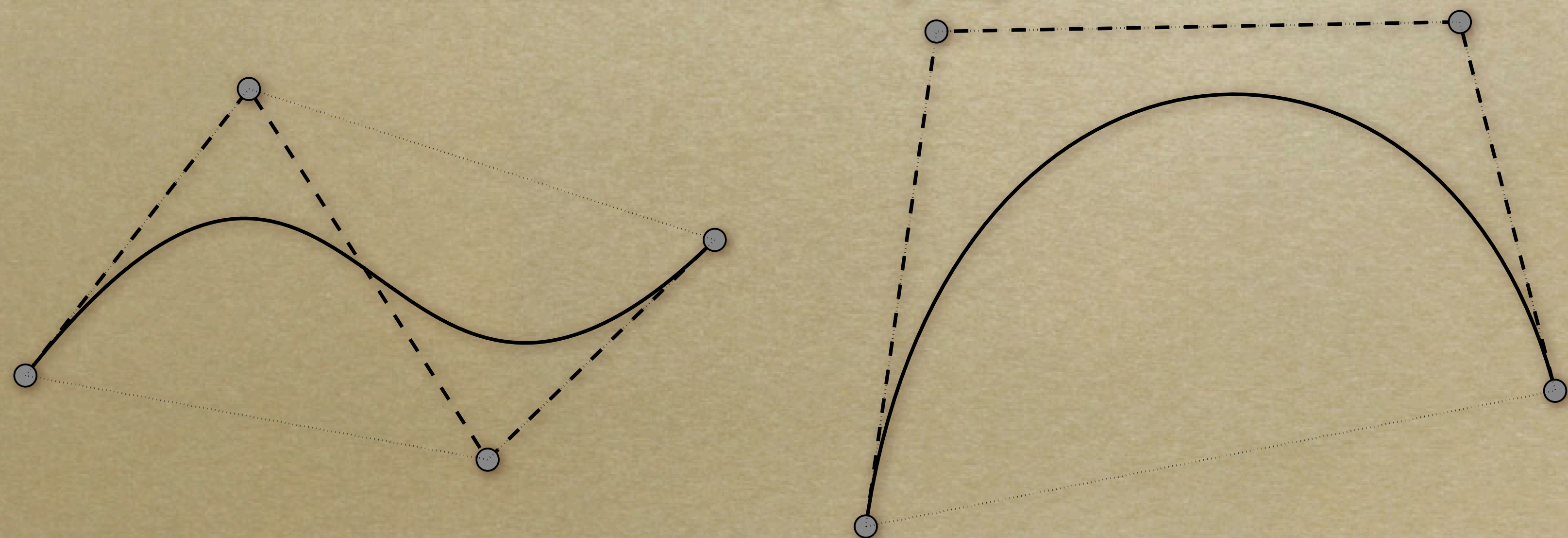
$$\mathbf{c} = \boldsymbol{\beta}_{\mathrm{Z}}\,\mathbf{p}_{\mathrm{Z}}$$

$$\mathbf{c} = \boldsymbol{\beta}_{\mathrm{H}}\,\mathbf{p}_{\mathrm{H}}$$

$$\mathbf{p}_{\mathrm{Z}} = \boldsymbol{\beta}_{\mathrm{Z}}^{-1}\,\boldsymbol{\beta}_{\mathrm{H}}\,\mathbf{p}_{\mathrm{H}}$$

Useful Properties of a Basis

- Convex Hull
  - All points on curve inside convex hull of control points
    - $\sum_i b_i(u) = 1 \qquad b_i(u) \geq 0 \qquad \forall u \in \Omega$
    - Bézier basis has convex hull property



Useful Properties of a Basis

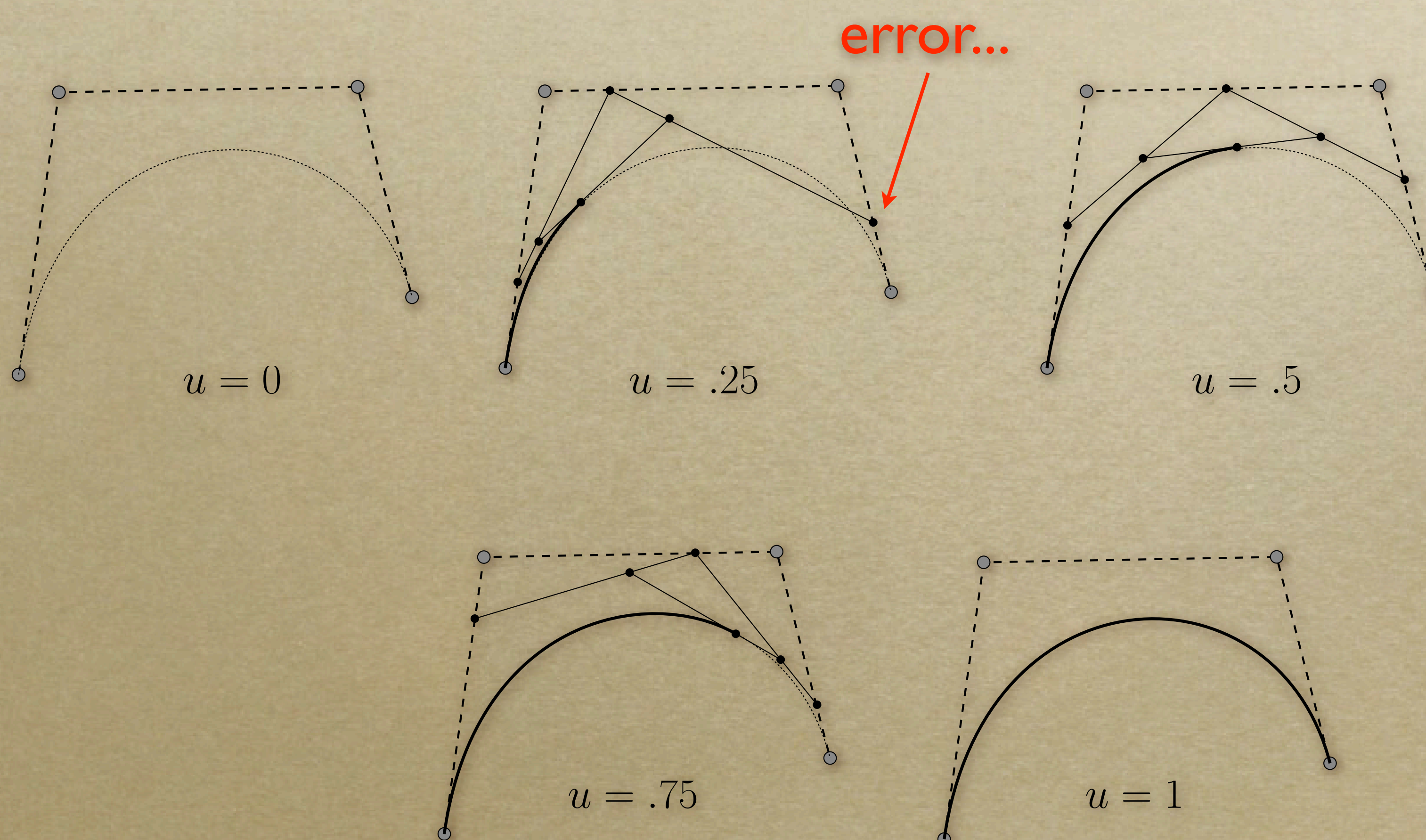
- Invariance under class of transforms
  - Transforming curve is same as transforming control points
    - $\boldsymbol{x}(u) = \sum_i \boldsymbol{p}_i b_i(u) \Leftrightarrow \mathcal{T}\boldsymbol{x}(u) = \sum_i (\mathcal{T}\boldsymbol{p}_i) b_i(u)$
    - Bézier basis invariant for affine transforms
    - Bézier basis NOT invariant for perspective transforms
      - NURBS are though...

Useful Properties of a Basis

- Local support
  - Changing one control point has limited impact on entire curve
  - Nice subdivision rules
  - Orthogonality ($\int_\Omega b_i(u) b_j(u) \mathrm{d}u = \delta_{ij}$)
  - Fast evaluation scheme
  - Interpolation -vs- approximation

DeCasteljau Evaluation

- A geometric evaluation scheme for Bézier



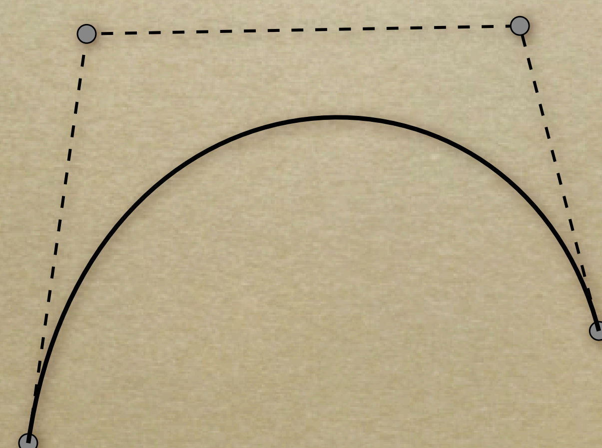
Adaptive Tessellation

- Midpoint test subdivision
- Possible problem
  - Simple solution if curve basis has *convex hull* property

If curve inside convex hull and the convex hull is nearly flat: curve is nearly flat and can be drawn as straight line

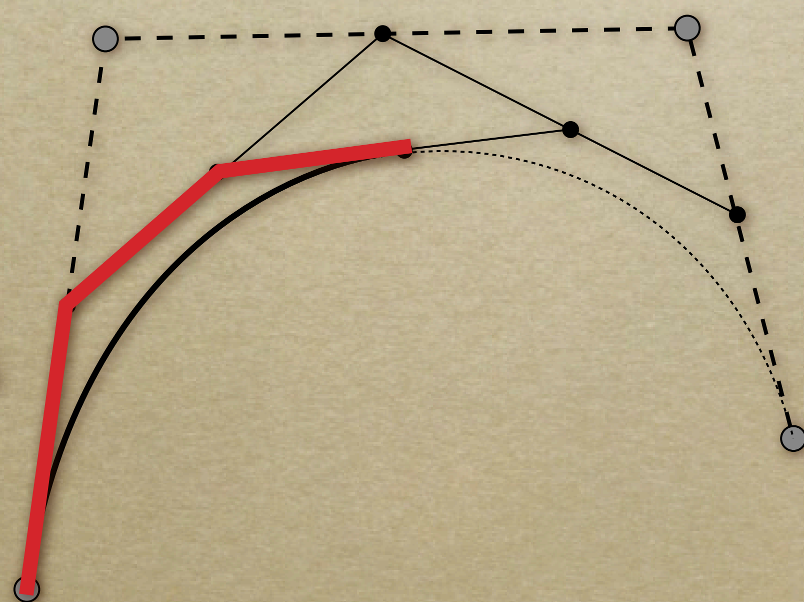
Better: draw convex hull

Works for Bézier because the ends are interpolated



Bézier Subdivision

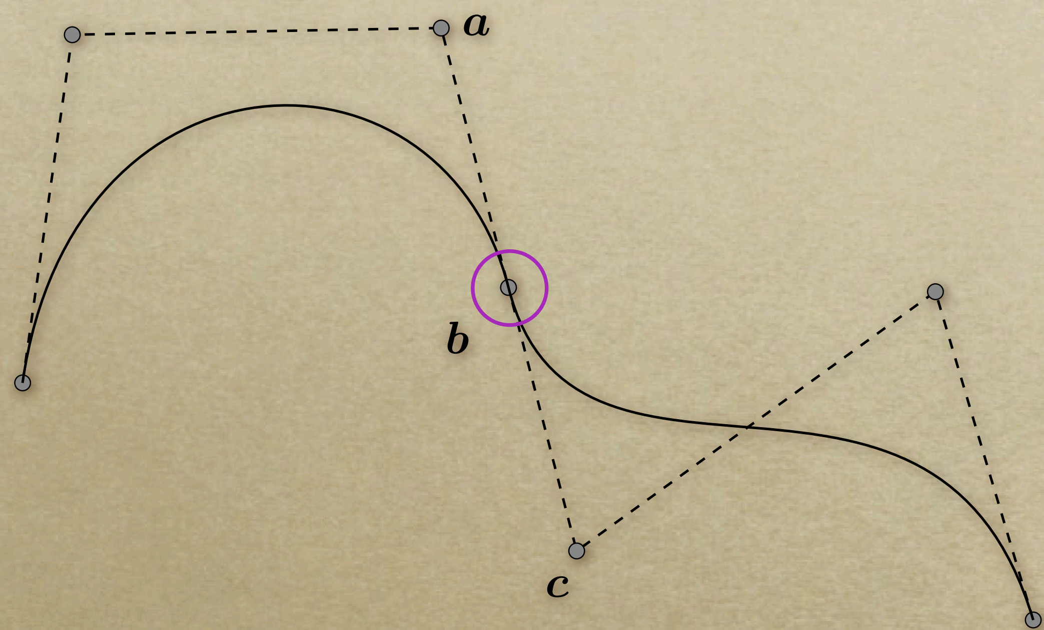
- Form control polygon for half of curve by evaluating at u=0.5



Repeated subdivision makes smaller/flatter segments

Also works for surfaces...

Joining



$$\mathcal{C}^0 \Leftrightarrow \boldsymbol{b} = \boldsymbol{b}$$

$$\mathcal{C}^1 \Leftrightarrow \boldsymbol{b} - \boldsymbol{a} = \boldsymbol{c} - \boldsymbol{b}$$

$$\mathcal{G}^1 \Leftrightarrow \frac{\boldsymbol{b} - \boldsymbol{a}}{||\boldsymbol{b} - \boldsymbol{a}||} = \frac{\boldsymbol{c} - \boldsymbol{b}}{||\boldsymbol{c} - \boldsymbol{b}||}$$

If you change $\boldsymbol{a}$, $\boldsymbol{b}$, or $\boldsymbol{c}$ you must change the others

But if you change $\boldsymbol{a}$, $\boldsymbol{b}$, or $\boldsymbol{c}$ you do not have to change beyond those three. *LOCAL SUPPORT*

"Hump" Functions

- Constraints at joining can be built in to make new basis

Tensor-Product Surfaces

- Surface is a curve swept through space
- Replace control points of curve with other curves

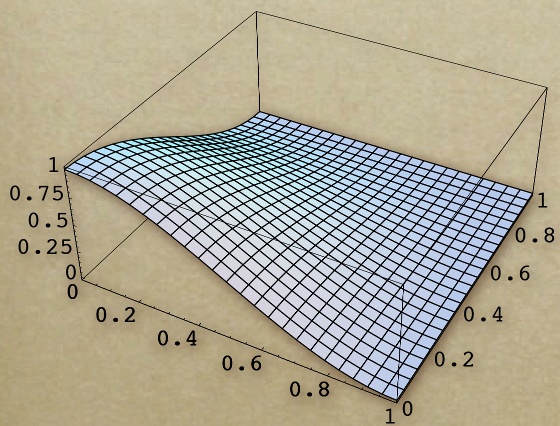
$$x(u,v) = \Sigma_i \, p_i \, b_i(u)$$

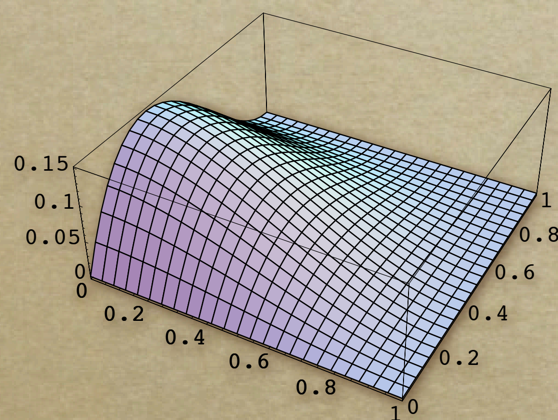
$$\Sigma_i \, q_i(v) \, b_i(u) \qquad\qquad q_i(v) = \Sigma_j \, p_{ji} \, b_j(v)$$

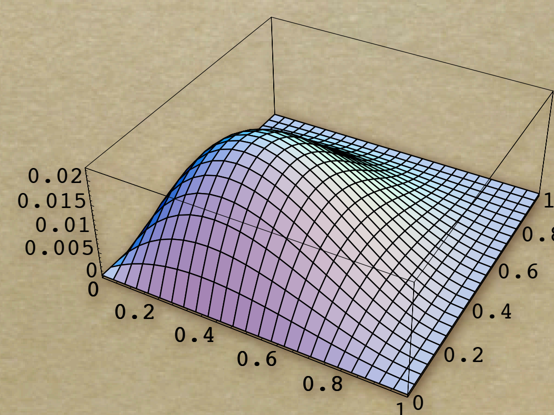
$$x(u,v) = \sum_{ij} p_{ij} b_i(u) b_j(v) \qquad\qquad b_{ij}(u,v) = b_i(u) b_j(v)$$

$$x(u,v) = \sum_{ij} p_{ij} b_{ij}(u,v)$$

Hermite Surface Bases

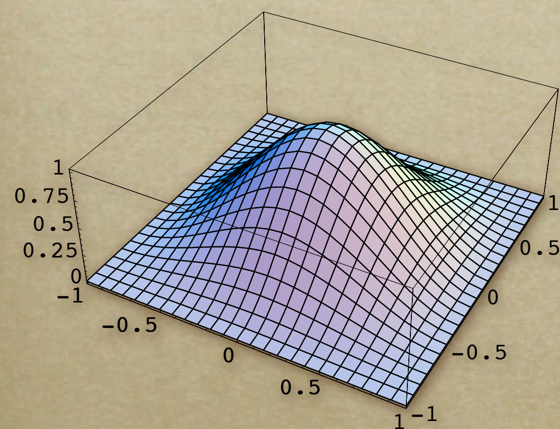


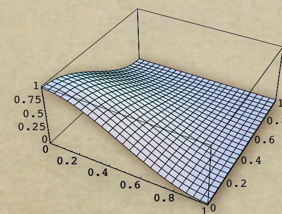


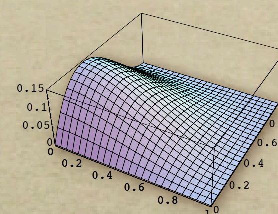


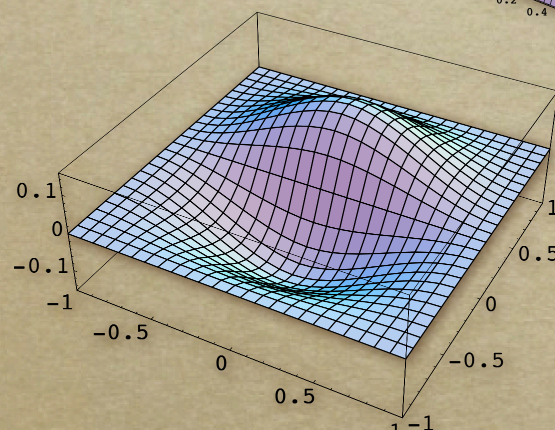
Plus symmetries...

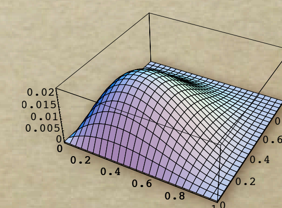
Hermite Surface Hump Functions

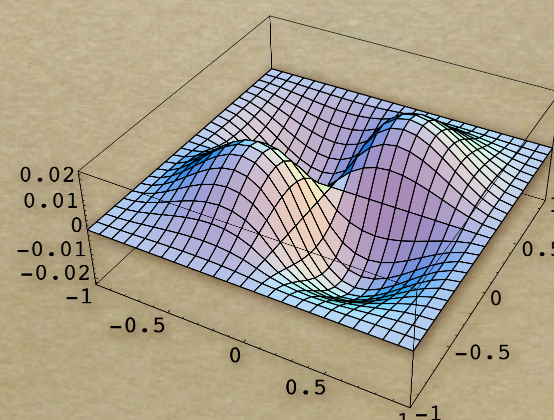








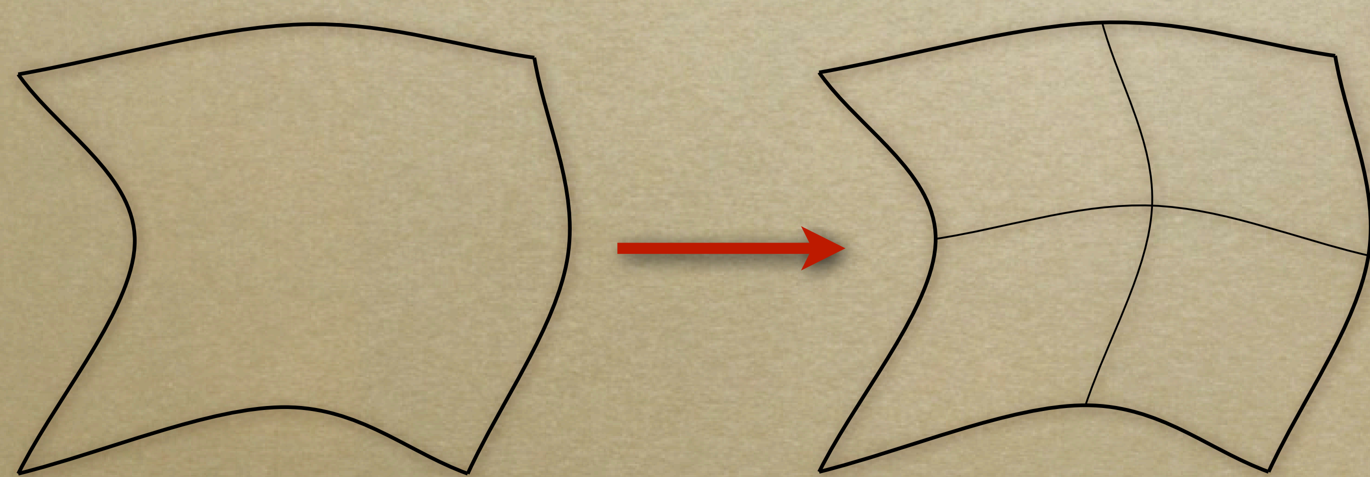




Plus symmetries...

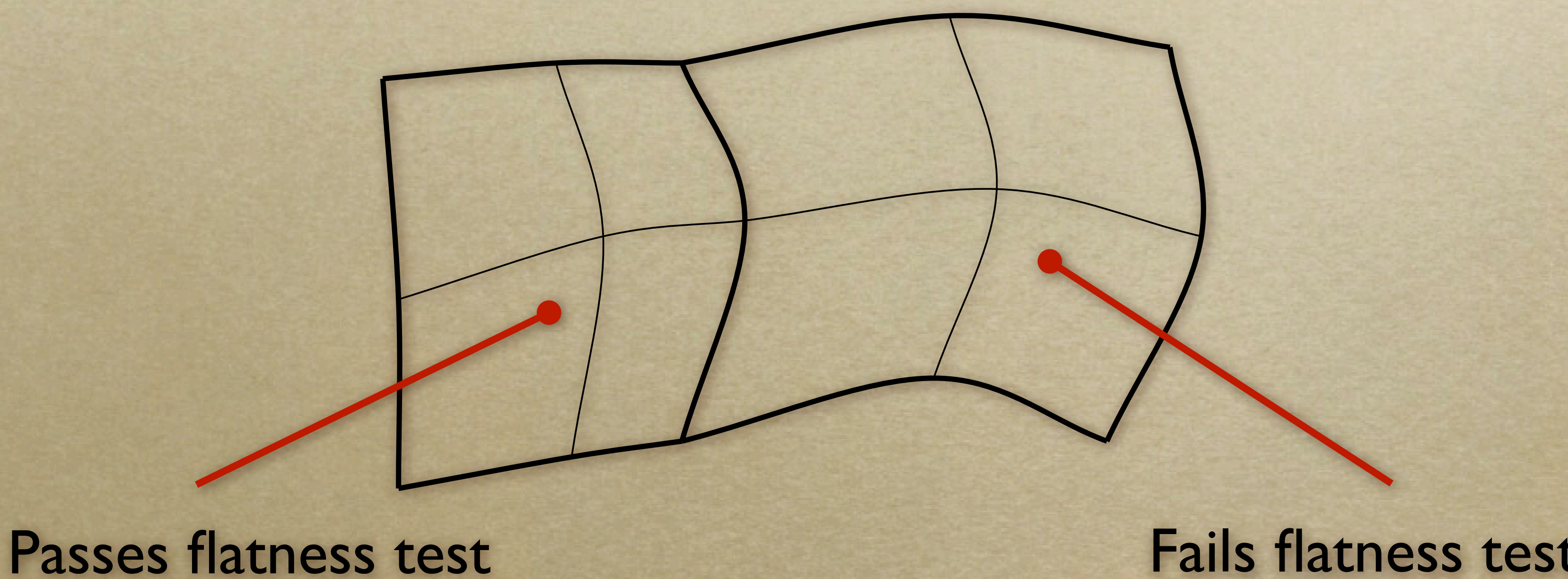
Adaptive Tessellation

- Given surface patch
  - If close to flat: draw it
  - Else subdivide 4 ways



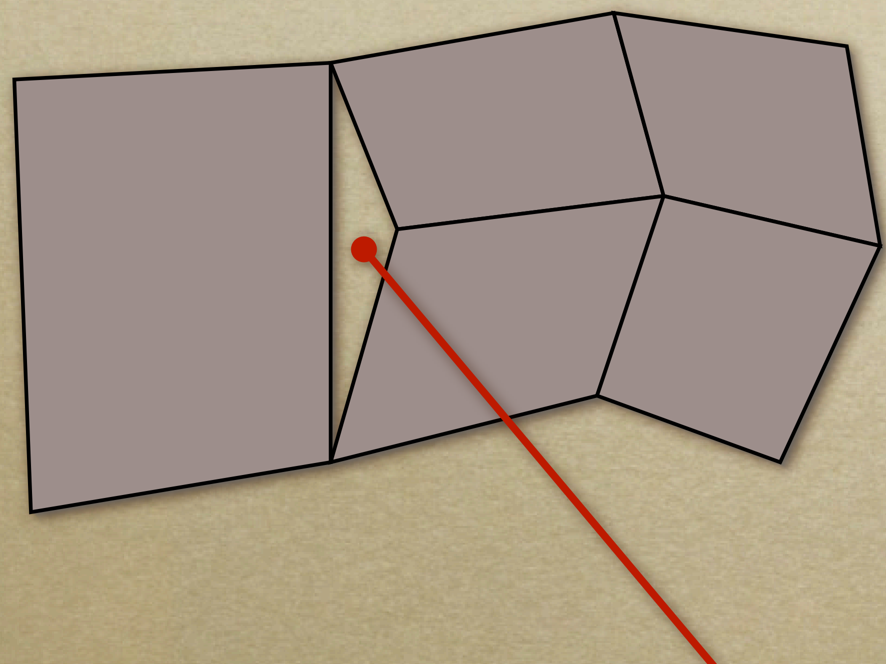
Adaptive Tessellation

- Avoid cracking



Adaptive Tessellation

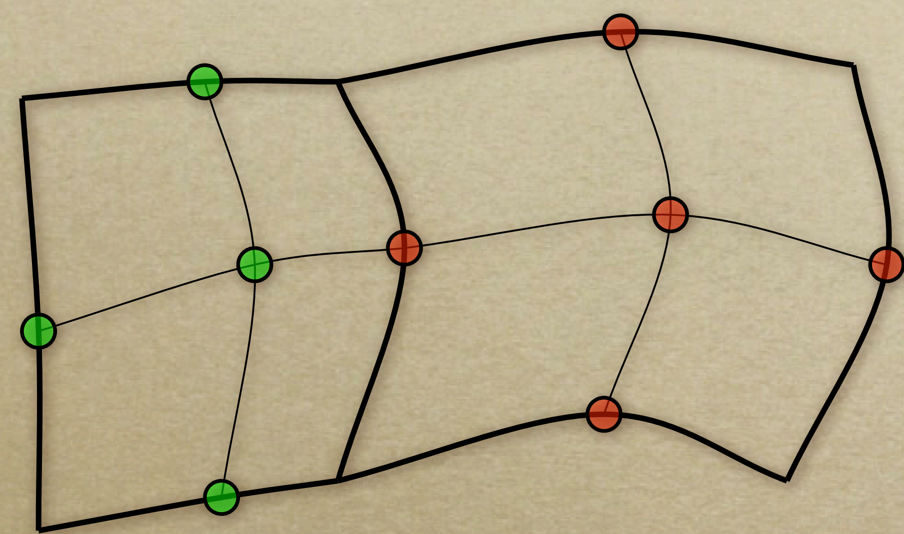
- Avoid cracking



Cracks may be okay in some contexts...

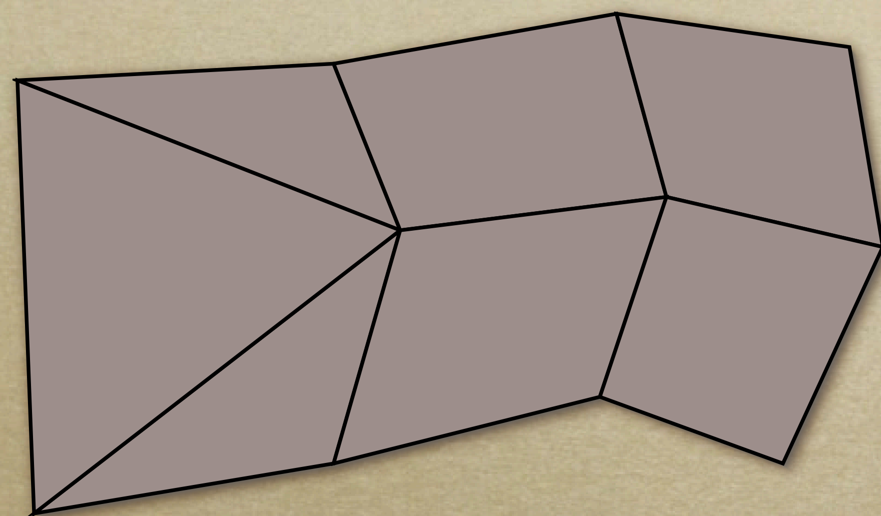
Adaptive Tessellation

- Avoid cracking



Adaptive Tessellation

- Avoid cracking



Test interior and boundary of patch
Split boundary based on boundary test
Table of polygon patterns
May wish to avoid “slivers”